Unit-II Central Difference formula's

1. Gauss's forward central difference interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2}$$
$$+ \cdots$$

2. Gauss's Backward central difference interpolation formula

$$y = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-3}$$
$$+ \cdots$$

3. Stirling's central difference interpolation formula

$$y = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$
$$+ \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \cdots$$

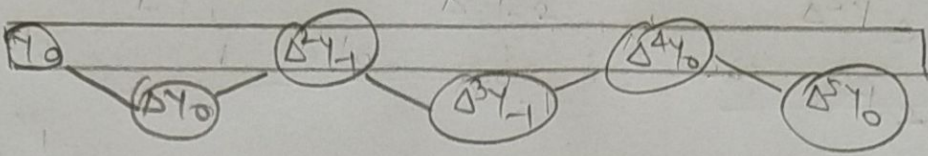
4. Bessel's central difference interpolation formula

$$y = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right]$$
$$+ \frac{u\left(u-\frac{1}{2}\right)(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right]$$
$$+ \frac{(u+1)u\left(u-\frac{1}{2}\right)(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \cdots$$

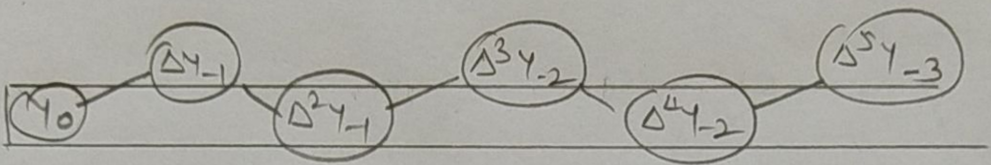
where $u = \dfrac{x - x_0}{h}$

1. G.F.

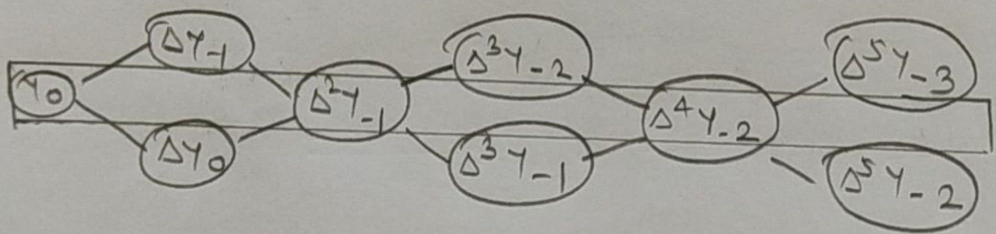
y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$



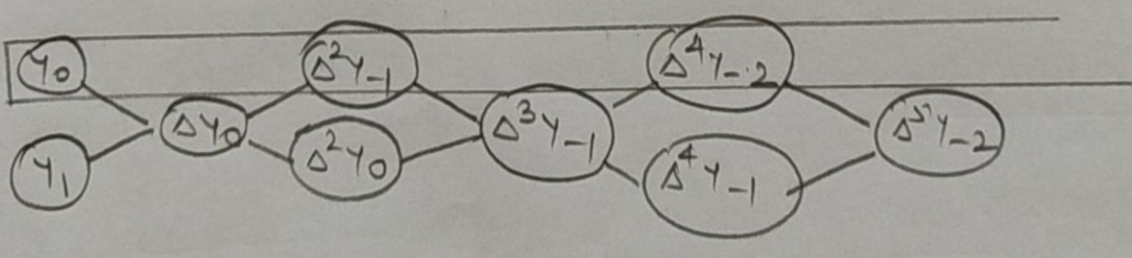
2. G.B



3. Stirlling



4. Bessel's



Q: Given that $f(20)=14$, $f(24)=32$, $f(28)=35$, $f(32)=49$ show by central difference interpolation formula's that ~~f(25)=~~ $f(25) = 33$.

Sol: Given that $f(20)=14$, $f(24)=32$, $f(28)=35$, $f(32)=40$

To find $f(25)=33$ Construct differen table

(i) for Gauss's forward difference interpalation formula

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^3 y_{-1} + \cdots$$

where $y = f(x)$, $u = \frac{x - x_0}{h}$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	20	14			
			18		
0	24	32		-15	
			3		17
1	28	35		2	
			5		
2	32	40			

$x = 25$, $x_0 = 24$, $u = \frac{25-24}{4} = \frac{1}{4}$

$$f(25) = 32 + \frac{1}{4}\times 3 + \frac{1}{2}\times\left(\frac{1}{4}-1\right)\times(-15) + \frac{1}{6}\times\frac{1}{4}\times\left(\frac{1}{16}-1\right)\times 17$$

$$= 32 + 0.75 + 1.40625 - 0.6640625$$

$$= 33.4921825$$

$$\boxed{f(25) \cong 33}$$

(ii) for Gauss's backward difference interpolation formula

$$y_u = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^3 y_{-2} + \cdots$$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-2	20	14			
			18		
-1	24	32		-15	
			3		17
0	28	35		2	
			5		
1	32	40			

$x = 25, \; x_0 = 28, \; h = 4, \quad u = \frac{25-28}{4} = -\frac{3}{4}$

$$f(25) = 35 - \frac{3}{4} \times 3 + \frac{1}{2}\left(-\frac{3}{4}\right)\left(-\frac{3}{4}+1\right) \times 2 + \frac{1}{6}\left(-\frac{3}{4}\right)\left(\frac{9}{16}-1\right) \times 17$$

$$= 35 - 2.25 - 0.1875 + 0.9296875$$

$$= 33.4921875$$

$\boxed{f(25) \simeq 33}$

(iii) for stirling's interpolation formula

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots$$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	20	14			
			18		
0	24	32		-15	
			3		17
1	28	35		2	
			5		
2	32	40			

$x = 25, \; x_0 = 24, \; h = 4, \quad u = \frac{25-24}{4} = \frac{1}{4}$

$$f(25) = 32 + \frac{1}{4} \times \left(\frac{18+3}{2}\right) + \frac{1}{2} \times \frac{1}{16} \times (-15) + \frac{1}{6} \times \frac{1}{4} \times \left(\frac{1}{16}-1\right)(17)$$

$$= 32 + 2.625 - 0.46875 - 0.6640625$$

$$= 33.4921875$$

$\boxed{f(25) \simeq 33}$

(iv) for Bessal's interpolatation formula

$$y_u = \left(\frac{y_0+y_1}{2}\right) + \cancel{u\Delta y_0} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)$$

$$+ \frac{u(u-\frac{1}{2})(u-1)}{3!}\Delta^3 y_{-1} + \cdots$$

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$\cancel{0}$ -1	20	14			
			18		
0	24	32		-15	
			3		17
1	28	35		2	
			5		
2	32	40			

$x = 25,\ x_0 = 24,\ h = 4,\quad u = \frac{25-24}{4} = \frac{1}{4}$

$$f(25) = \left(\frac{32+35}{2}\right) + \left(\frac{1}{4} - \frac{1}{2}\right)\times 3 + \frac{1}{2}\times\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{-15+2}{2}\right)$$

$$+ \frac{1}{6}\times\frac{1}{4}\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{1}{4}-1\right)\times 17$$

$= 33.5 - 0.75 + 0.609375 + 0.1328125$

$= 33.4921875$

$\boxed{f(25) \simeq 33}$

∴ ~~By all four~~

∵ By any formula we get

$f(25) = 33.4921875 \simeq 33$ (Approx)

∴ Here $f(25) = 33$.

2. गॉस अग्र अन्तर्वेशन सूत्र से y_{30} ज्ञात कीजिए, जहाँ दिया है:

Use Gauss's forward interpolation formula to find the value of $f(x)$ for $x = 30$, given that:

[Jodhpur 18

x	:	21	25	29	33	37
$f(x)$	:	18.4708	17.8144	17.1070	16.3432	15.5154

3. गॉस अन्तर्वेशन सूत्र द्वारा निम्न ऑकडों से $f(41)$ का मान ज्ञात कीजिए.

Use Gauss's interpolation formula to find $f(41)$ with the help of the following data:

x	:	30	35	40	45	50
$f(x)$	:	3678.2	2995.1	2400.1	1876.2	1416.3

4. निम्न सारणी से गॉस के अग्र सूत्र द्वारा $x = 3.75$ पर y का मान ज्ञात कीजिए.

Use Gauss's forward formula to find the value of y when x=3.75 from the following table

x:	2.5	3.0	3.5	4.0	4.5	5.0
y:	24.145	22.043	20.225	18.644	17.262	16.047

[Jodhpur 04; Bikaner 04, 17, 18; Raj. 15, 17]

5. यदि $f(20)=14, f(24)=32, f(28)=35$ तथा $f(32)=40$ हो, तो गॉस के सूत्र द्वारा प्रदर्शित कीजिए कि $f(25)=33$ होगा।

Given that $f(20)=14, f(24)=32, f(28)=35, f(32)=40$, show by Gauss's formula that $f(25)=33$

6. गॉस के अग्रगामी तथा पश्चगामी केन्द्रीय अन्तर अन्तर्वेशन सूत्रों को प्रतिपादित कीजिये। इनसे अथवा किसी अन्य विधि से स्टरलिंग का अन्तर्वेशन सूत्र प्राप्त कीजिये। इन सूत्रों के मार्गों को अन्तर सारणी में प्रदर्शित कीजिये।

Derive Gauss-forward and backward interpolation formulae of Central differences.Hence or otherwise obtain striling's interpolation formula. Show the paths of these formulae in difference table.

7. गॉस पश्च सूत्र द्वारा 1936 की जनसंख्या ज्ञात कीजिए जहाँ दिया है

Use Gauss's Backward formula to find the population in the year 1936. Given that

वर्ष(Year):	1901	1911	1921	1931	1941	1951
जनसंख्या:	12	15	20	27	39	52

हजारों में (Population in Thousand) [Alwar 17]

8. दिया हुआ है (Given that)

$\sqrt{(12500)} = 111.803399$; $\sqrt{(12510)} = 111.848111$;

$\sqrt{(12520)} = 111.892806$; $\sqrt{(12530)} = 111.937483$;

गॉस पश्च सूत्र से प्रदर्शित कीजिए कि (Show by Gauss's backward formula that):

$$\sqrt{(12516)} = 111.874930;$$ [Ajmer 03]

9. स्टरलिंग और बेसल के अन्तर्वेशन सूत्रों को समझाइये और तुलना कीजिए। Discuss and compare Stirling's and Bessel's formulae for interpolation

10. स्टरलिंग के अन्तर्वेशन सूत्र द्वारा निम्नलिखित सारणी से $y(1.4171)$ का मान ज्ञात कीजिए।

Use Stirling's interpolation formula to compute $y(1.4171)$ from the following table:

x:	1.0	1.1	1.2	1.3	1.4
y:	1.1752	1.3357	1.5095	1.6984	1.9043
x:	1.5	1.6	1.7	1.8	
y:	2.1293	2.3756	2.6451	2.9422	

11. स्टरलिंग के सूत्र द्वारा निम्न सारणी से $f(1.22)$ का मान ज्ञात कीजिए :
Use Stirling's formula to find $f(1.22)$ from the following table :

x:	1.0	1.1	1.2	1.3	1.4
$f(x)$:	0.84147	0.89121	0.93204	0.96356	0.98545
x:	1.5	1.6	1.7	1.8	
$f(x)$:	0.99749	0.99957	0.97385	0.97385	

12. स्टरलिंग सूत्र द्वारा निम्न ऑंकडो से y_{35} ज्ञात कीजिए।
Use Stirling's formula to find y_{35}, given

$y_{20} = 512, y_{30} = 439, y_{40} = 346, y_{50} = 243$ [Sikar 17]

जहाँ वय सारणी में x वर्ष की आयु पर व्यक्तियों की संख्या को y_x निरुपित करता है। Where y_x represents the number of persons at the age x years in a life table. Also use Bessel's formula. [Raj. 14, 16; Bikaner 05]

13. दिया है (Given)

$\theta =$	0^0	5^0	10^0	15^0	20^0	25^0	30^0
$\tan\theta =$	0.0000	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

स्टरलिंग सूत्र द्वारा $\tan 16^0$ का मान ज्ञात कीजिए।

Use Stirling's formula to find the value of $\tan 16^0$ [Udaipur 07, 17]

14. स्टरलिंग के सूत्र से निम्न आंकड़ों द्वारा u_{32} का मान ज्ञात कीजिए :

Use Stirling's formula to find u_{32} from the following data :

$u_{20} = 14.035$ $\quad u_{25} = 13.674$ $\quad u_{30} = 13.257$

$u_{35} = 12.734$ $\quad u_{40} = 12.089$ $\quad u_{45} = 11.309$ [Ajmer 05,08]

15. स्टरलिंग सूत्र द्वारा ऑकडो से y_{11} ज्ञात कीजिए :

Use Stirling's formula to find y_{11} from the following data :

$y_0 = 3010, y_5 = 2710, y_{10} = 2285, y_{15} = 1860, y_{20} = 1560,$

$y_{25} = 1510, y_{30} = 1835$ [Ajmer 15]

16. केन्द्रीय अन्तरों से क्या तात्पर्य होता है?

What is meant by central differences?

17. बेसल अन्तर्वेशन सूत्र प्राप्त कीजिए और उसकी सहायता से $\cos 23^0$ का मान ज्ञात कीजिए।

Obtain Bessel's interpolation formula and use it to find the value of $\cos 23^0$.

18. बेसल सूत्र द्वारा निम्न ऑकडों से y_{25} ज्ञात कीजिए:

Use Bessel's formula to find y_{25} from the following data :

$y_{20} = 2854;$ $\quad y_{24} = 3162;$ $\quad y_{28} = 3544;$ $\quad y_{32} = 3922;$

[Raj. 17; Ajmer 06, 08]

19. बेसल सूत्र (अथवा गॉस सूत्र) द्वारा $x = 3.75$ के लिए निम्न सारणी से y का मान ज्ञात कीजिए:

Use Bessel's formula (or Gauss formula) to find y for $x = 3.75$ from the following table:

x:	2.5	3.0	3.5	4.0	4.5	5.0
y:	24.145	22.043	20.225	18.644	17.262	16.047

[Ajmer 09]

20. अन्तराल 0.5 पर $t=0$ से $t=3$ तक $f(t)$ के मान निम्न सारणी में दिए हैं जहाँ $f(t)$ निम्न समाकलन से व्यक्त है:

From the following table which gives the values of $f(t)$ at interval of $t=0.5$ from $t=0$ to $t=3$, where $f(t)$ represents the integral

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_0^t e^{-t^2/2}dt$$

t :	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$f(t)$:	0.000	0.191	0.341	0.433	0.477	0.494	0.499

(i)गॉस सूत्र, (ii)स्टरलिंग सूत्र तथा (iii)बेसल सूत्र द्वारा $f(1.22)$ का अनुमानित मान ज्ञात कीजिए।

Estimate the vale of $f(1.22)$ by (i) Gauss formula, (ii) Striling formula and (iii) Bessel Formula. **[Bikaner 08; Ajmer 07]**

21. निम्नलिखित सारणी में दोनों (बेसल व स्टरलिंग) सूत्रों द्वारा अलग–अलग y(1.7475) का मान ज्ञात कीजिए :

Evaluate y(1.7475) from the following table by both Bessel and Stirling's formulae separately:

x :	1.72	1.73	1.74	1.75
y :	0.179066	0.177284	0.175520	0.173774
x :	1.76	1.77	1.78	
y :	0.172044	0.170333	0.168638	

22. निम्न सारणी से $f(0.5437)$ का मान गॉस, स्टरलिंग तथा बेसल सूत्र से ज्ञात कीजिए:

From the following table find the value of $f(0.5437)$ by Gauss, Stirling and Bessel formulae:

x :	0.51	0.52	0.53	0.54
$f(x)$:	0.5292437	0.5237899	0.5464641	0.5549392
x :	0.55	0.56	0.57	
$f(x)$:	0.5633233	0.5716157	0.5798158	

23. गॉस सूत्र को लगाकर त्रिघात बहुपद ज्ञात कीजिए जो y को नीचे दिए मानों को ग्रहण करता है:

Apply Gauss's formula to find a polynomial of degree three which takes the value of y as given below:

[Ajmer 07]

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
2	−2	−2				
			3			
4	−1	1		−1		
			2		4	
6	0	3		3		0
			5		4	
8	1	8		7		
			12			
10	2	20				

24. सिद्ध कीजिए (Prove that):

$$\mu^{-1} = 1 - \frac{1}{8}\delta^2 + \frac{3}{128}\delta^4 - ....$$

[Ajmer 07]

25. निम्नलिखित सारणी से $f(27.4)$ का मान बेसल सूत्र से ज्ञात कीजिए:

Use Bessel's formula to find $f(27.4)$ from the following table:

x:	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

[Ajmer 14]

26. स्टिरलिंग सूत्र का प्रयोग करके u_{32} का मान ज्ञात कीजिए:

Use Stirling's formula to find u_{32} from the following table:

x:	20	25	30	35	40	45
u_x:	14.035	13.674	13.257	12.734	12.089	11.309

[Ajmer 17]

27. निम्न सारणी से $e^{1.44}$ का मान स्टिरलिंग विधि से ज्ञात कीजिए:

Use Stirling's formula to find $e^{1.44}$ from the following table :

x:	1.00	1.20	1.40	1.60	1.80	2.00
e^x:	2.7183	3.3201	4.0552	4 9530	6.0496	7.3891

[Raj. 18]

□□□

प्रश्नावली (Exercise) IV

2.	16.9216	3.	2290.1
4.	19.407	7.	32.3437 हजार
10.	1.94142	11.	0.9340
12.	395 (लगभग)	13.	0.2867
14.	13.060862	15.	2196
18.	3250.875	19.	19.407

20. (i) 0.389 (ii) 0.3886 (iii) 0.388

21. 0.1739652; 0.173962

22. (i), (ii) & (iii) 0.558052 (लगभग)

23. $y_u = \frac{1}{6}[4u^3 + 9u^2 + 17u + 18], u = \frac{x-6}{2}$

25. 3.6497

(8)

u	x	$f(x)=\sqrt{x}$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-2	12500	111.803399			
			0.044712		
-1	12510	111.848111		-0.000017	
			0.044695		-0.000001
0	12520	111.892806		-0.000018	
			0.044677		
1	12530	111.937483			

Gauss backward

$$f(x)=y_u=y_0+u\Delta y_{-1}+\frac{u(u+1)}{2!}\Delta^2 y_{-1}+\frac{(u-1)u(u+1)}{3!}\Delta^3 y_{-2}$$

where $u=\dfrac{x-x_0}{h}=\dfrac{12516-12520}{10}=-0.4$

$$f(12516)=111.892806+(-0.4)\times 0.044695+\frac{(-0.4)\times(0.6)\times(-0.000018)}{2}+\frac{(-1.4)\times(-0.4)(0.6)}{6}\times 0.000001$$

$$=111.892806-0.017878+0.00000216-0.000000056$$

$$\sqrt{12516}=111.8749302$$
$$\simeq 111.874930 \text{ (Approx)}$$

(23)

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	2	-2			
			3		
				-1	
-1	4	1			4
			2		
0	6	3		3	
			5		4
	8	8		7	
1			12		
2	10	20			

Gauss forward: $f(x)=y_0+u\Delta y_0+\dfrac{u(u-1)}{2!}\Delta^2 y_{-1}+\dfrac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$

where $u=\dfrac{x-6}{2}$

$$f(x) = 3 + \frac{(x-6)}{2} \times 5 + \frac{\left(\frac{x-6}{2}\right) \times \left(\frac{x-6}{2} - 1\right)}{2} \times 3$$

$$+ \frac{\left(\frac{x-6}{2}\right)\left(\frac{x-6}{2} - 1\right)\left(\frac{x-6}{2} + 1\right)}{6} \times 4$$

$$= 3 + \frac{5}{2}(x-6) + \frac{3}{2}\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)$$

$$+ \frac{2}{3}\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)\left(\frac{x-4}{2}\right)$$

$$= 3 + \frac{5}{2}x - 15 + \frac{3}{8}(x^2 - 14x + 48) + \frac{1}{12}(x^3 - 18x^2 + 104x - 192)$$

$$= \frac{1}{24}\left[-288 + 60x + 9(x^2 - 14x + 48) + 2(x^3 - 18x^2 + 104x - 192)\right]$$

$$= \frac{1}{24}\left[-288 + 60x + 9x^2 - 126x + 432 + 2x^3 - 36x^2 + 208x - 384\right]$$

$$f(x) = \frac{1}{24}\left[2x^3 - 27x^2 + 142x - 240\right]$$

for in term of u

$\because u = \frac{x-6}{2} \Rightarrow x = 2u + 6$

$$y_u = \frac{1}{24}\left[2(2u+6)^3 - 27(2u+6)^2 + 142(2u+6) - 240\right]$$

$$= \frac{1}{24}\left[2(8u^3 + 72u^2 + 216u - 216) - 27(4u^2 + 24u + 36) + 284u + 852 - 240\right]$$

$$= \frac{1}{24}\left[16u^3 + 36u^2 + 68u + 72\right]$$

$$= \frac{1}{6}\left[4u^3 + 9u^2 + 17u + 18\right]$$

Central Difference operater (δ):

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\Rightarrow \boxed{\delta \equiv E^{1/2} - E^{-1/2}}$$

Average operator (μ):

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$\Rightarrow \boxed{\mu \equiv \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)}$$

Central Sum Operater (σ):

$$f(x) = \sigma\left[f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right]$$

$$\Rightarrow \boxed{\sigma\delta \equiv 1}$$

but $\sigma\delta \not\equiv \delta\sigma$

Operator Relations

(i) $\delta \equiv \Delta E^{-1/2} \equiv \Delta(1+\Delta)^{-1/2} \equiv \nabla E^{1/2}$

(ii) $\mu^2 \equiv \frac{1}{4}(\delta^2+4)$

(iii) $\mu\delta \equiv \frac{1}{2}(\Delta+\nabla)$

(iv) $\sigma \equiv \dfrac{E^{1/2}}{E-1}$

(v) $\Delta \equiv \frac{1}{2}\delta^2 + \delta\sqrt{\left(1+\frac{\delta^2}{4}\right)}$

(vi) $\delta^2 \equiv \Delta\nabla \equiv \Delta - \nabla$

(vii) $(E+1)\delta \equiv 2(E-1)\mu$

(viii) $\sqrt{(1+\delta^2\mu^2)} \equiv 1 + \frac{1}{2}\delta^2$

(ix) $\Delta + \nabla \equiv \frac{\nabla}{\Delta} - \frac{\Delta}{\nabla}$

(x) $E^{-1/2} \equiv \mu - \frac{1}{2}\delta$

(xi) $\mu \equiv \dfrac{2-\nabla}{2\sqrt{1-\nabla}}$

(i) $\delta \equiv \Delta E^{-1/2} \equiv \nabla E^{1/2}$

$\because \delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$

$= E^{1/2} f(x) - E^{-1/2} f(x)$

$\therefore \delta \equiv E^{1/2} - E^{-1/2}$

$\equiv E^{1/2}(1 - E^{-1})$

$\boxed{\delta \equiv E^{1/2}\nabla}$

or

$\delta \equiv E^{-1/2}(E-1) \Rightarrow \boxed{\delta \equiv E^{-1/2}\Delta}$

$\equiv \cancel{\delta}^{-1/2} E$

(ii) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

$\because \mu \equiv \frac{1}{2}(E^{1/2} + E^{-1/2})$

$\therefore \mu^2 \equiv \frac{1}{4}(E + E^{-1} + 2)$

$\equiv \frac{1}{4}\left[(E^{1/2} - E^{-1/2})^2 + 4\right]$

$\Rightarrow \boxed{\mu^2 \equiv 1 + \frac{1}{4}\delta^2}$

(iii) $\mu\delta \equiv \frac{1}{2}(\Delta + \nabla)$

$\because \mu \equiv \frac{1}{2}(E^{1/2} + E^{-1/2})$ & $\delta \equiv E^{1/2} - E^{-1/2}$

$\therefore \mu\delta \equiv \frac{1}{2}(E - E^{-1})$

$\equiv \frac{1}{2}\left[\frac{1}{E}(E^2 - 1)\right]$

$\equiv \frac{1}{2}\left[\frac{1}{E}(E-1)(E+1)\right]$

$\equiv \frac{1}{2}\left[\Delta(1 + E^{-1})\right] \quad \because E - I \equiv \Delta$

$\equiv \frac{1}{2}(\Delta + \Delta E^{-1}) \quad \because \Delta E^{-1} \equiv \nabla$

$\boxed{\mu\delta \equiv \frac{1}{2}(\Delta + \nabla)}$

(iv) $\sqrt{(1+\delta^2\mu^2)} \equiv 1+\frac{1}{2}\delta^2$

$\because \mu \equiv \frac{1}{2}(E^{1/2}+E^{-1/2})$ & $\delta \equiv E^{1/2}-E^{-1/2}$

$\therefore \delta\mu \equiv \frac{1}{2}(E-E^{-1})$

$\Rightarrow 1+\delta^2\mu^2 \equiv 1+\frac{1}{4}(E-E^{-1})^2$

$$\equiv \frac{4+E^2+E^{-2}-2}{4}$$

$$\equiv \frac{(E+E^{-1})^2}{4}$$

$$\equiv \frac{\left[(E^{1/2}-E^{-1/2})^2+2\right]^2}{4}$$

$$\equiv \left[1+\frac{1}{2}\delta^2\right]^2$$

$\Rightarrow \boxed{\sqrt{(1+\delta^2\mu^2)} \equiv 1+\frac{1}{2}\delta^2}$

(v) $\mu \equiv \dfrac{2-\nabla}{2\sqrt{1-\nabla}}$

$\because \nabla \equiv 1-E^{-1}$

$\therefore \dfrac{2-\nabla}{2\sqrt{(1-\nabla)}} \equiv \dfrac{2-(1-E^{-1})}{2\sqrt{[1-(1-E^{-1})]}}$

$$\equiv \frac{1+E^{-1}}{2\sqrt{E^{-1}}}$$

$$\equiv \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right]$$

$\boxed{\dfrac{2-\nabla}{2\sqrt{1-\nabla}} \equiv \mu}$

24: $\mu^{-1} = 1 - \frac{1}{8}\delta^2 + \frac{3}{128}\delta^4 - \cdots$

Sol: $\because$ $\delta \equiv (E^{1/2} - E^{-1/2})$ & $\mu \equiv \frac{1}{2}(E^{1/2} + E^{-1/2})$

$$\mu^2 \equiv \frac{1}{4}\left[E + E^{-1} + 2\right]$$

$$\equiv \frac{1}{4}\left[(E^{1/2} - E^{-1/2})^2 + 4\right]$$

$$\mu^2 \equiv 1 + \frac{1}{4}\delta^2$$

$$\Rightarrow \mu \equiv \left(1 + \frac{1}{4}\delta^2\right)^{1/2}$$

$$\Rightarrow \mu^{-1} \equiv \left(1 + \frac{1}{4}\delta^2\right)^{-1/2}$$

by binomial theorem

$$\mu^{-1} \equiv 1 + \left(-\frac{1}{2}\right)\left(\frac{1}{4}\delta^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{1}{4}\delta^2\right)^2$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{6!}\left(\frac{1}{4}\delta^2\right)^3$$

$$\Rightarrow \mu^{-1} \equiv 1 - \frac{1}{8}\delta^2 + \frac{3}{128}\delta^4 - \cdots$$

Unit - II Numerical Differentiation

Newton's Gregory forward difference interpolation formula:

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \cdots$$

$$= f(a) + u\Delta f(a) + \frac{1}{2!}(u^2-u)\Delta^2 f(a) + \frac{1}{3!}(u^3-3u^2-2u)\Delta^3 f(a)$$

$$+ \frac{1}{4!}(u^4-6u^3+11u^2-6u)\Delta^4 f(a) + \text{—} \quad \text{— (1)}$$

where $u = \frac{x-a}{h}$ — (2)

$$\because f'(x) = \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

$\therefore$ by (1) & (2)

$$f'(x) = \frac{1}{h}\Big[\Delta f(a) + \frac{1}{2!}(2u-1)\Delta^2 f(a) + \frac{1}{3!}(3u^2-6u-2)\Delta^3 f(a)$$

$$+ \frac{1}{4!}(4u^3-18u^2+22u-6)\Delta^4 f(a) + \text{—}\Big]$$

Similarly

$$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{du}\left(\frac{df}{dx}\right) = \frac{d}{du}\left(\frac{df}{du}\right)\cdot\frac{du}{dx}$$

$$= \frac{1}{h^2}\Big[\Delta^2 f(a) + \frac{1}{3!}(6u-6)\Delta^3 f(a) + \frac{1}{4!}(12u^2-36u+22)\Delta^4 f(a) + \text{—}\Big]$$

Similarly we can find ~~first ds~~ Successive ~~diffe~~ derivative from all the interpolation formula's.

Q: find $f'(1)$ for $f(x) = \frac{1}{1+x^2}$ using the following data:

x :	1.0	1.1	1.2	1.3	1.4
$f(x)$:	0.5	0.4525	0.4098	0.3717	0.3378

<u>Solution</u>: To find $f'(1)$ for $f(x) = \frac{1}{1+x^2}$ from the given table we use Newton's gregory forward difference interpolation formula for this difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0.5				
		-0.0475			
1.1	0.4525		0.0048		
		-0.0427		-0.0002	
1.2	0.4098		0.0046		-0.0002
		-0.0381		-0.0004	
1.3	0.3717		0.0042		
		-0.0339			
1.4	0.3378				

Newton's formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + - \quad —$$

where $u = \frac{x-a}{h}$

$$f'(x) = \frac{df}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta f(a) + \frac{1}{2!}(2u-1)\Delta^2 f(a) + \frac{1}{3!}(3u^2-6u-2)\Delta^3 f(a)\right.$$

$$\left. + \frac{1}{4!}(4u^3 - 18u^2 + 22u - 6)\Delta^4 f(a) + -\right]$$

$\because x = 1,\ h = 0.1,\ a = 1$

$\therefore u = 0$

$$f'(1) = \frac{1}{0.1}\left[-0.0475 + \frac{1}{2}(-1)(0.0048) + \frac{1}{6}(-2)(-0.0002)\right.$$

$$\left. + \frac{1}{24}(-6)(-0.0002)\right]$$

$$= 10[-0.0475 - 0.0024 + 0.000067 + 0.00005]$$

$$= 10\times(-0.04983) = -0.4978 \text{ (Approx)}$$

verification $f'(1) = -0.4978 \approx -0.5$ (Approx).

$$f(x) = \frac{1}{1+x^2}$$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

$$f'(1) = \frac{-2}{(1+1^2)^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$\Rightarrow \boxed{f'(1) = -0.5}$$

Q: find the first two derivatives for $f(x)$ at $x=1$ from the following table:

x :	-2	-1	0	1	2	3	4
$f(x)$:	104	17	0	-1	8	69	272

Solution: To find the derivatives of $f(x)$ at $x=1$ from the given table, we use stirling's interpolation formula for this differen table

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-3	-2	104				
			-87			
				70		
-2	-1	17			-54	
			-17			48
-1	0	0		16		
			-1		-6	
0	1	-1		10		48
			9		42	
1	2	8		52		48
			61		90	
2	3	69		142		
			203			
3	4	272				

stirling formula

$$\cancel{f(x)}\ y = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{---}(1)$$

where $u = \frac{x - x_0}{h}$

$$f'(x) = y' = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{1}{6}(3u^2-1)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{24}(4u^3 - 2u)\Delta^4 y_{-2} + \text{---} \right] \quad \text{--- (2)}$$

$\because x = 1, \; x_0 = 1, \; h = 1$

$\therefore u = 0$

$$f'(1) = \frac{1}{1}\left[4 - \frac{1}{6} \times 18 \right]$$

$$\boxed{f'(1) = 1}$$

$$f''(x) = y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right) \cdot \frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{24}(12u^2 - 2)\Delta^4 y_{-2} + \text{--} \right]$$

$$= \frac{1}{1^2}\left[10 + 0 + \frac{1}{24}(-2)(48) \right]$$

$$= 10 - 4$$

$$\boxed{f''(1) = 6}$$

Note: You can verify your answer's by using another formula's.

Q: Find $f'(6)$ from the following table:

x :	0	1	3	4	5	7	9
$f(x)$:	150	108	0	-54	-100	-144	-84

Solution: ~~For find f'(x0)~~

$\because$ given data's are at unequal interval

$\therefore$ to find $f'(6)$, we use Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \cdots$$

$$f'(x) = \Delta f(x_0) + [(x-x_0)+(x-x_1)]\Delta^2 f(x_0)$$
$$+ [(x-x_0)(x-x_1) + (x-x_0)(x-x_2) + (x-x_1)(x-x_2)\Delta^3 f(x_0)$$
$$+ \cdots]$$

where

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x :	0	1	3	4	5	7	9
$f(x)$:	150	108	0	-54	-100	-144	-84

Divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	150			
		-42		
1	108		-4	
		-54		1
3	0		0	
		-54		1
4	-54		4	
		-46		1
5	-100		8	
		-22		1
7	-144		13	
		30		
9	-84			

$$f'(6) = -42 + [(6-0)+(6-1)](-4) + [(6-0)(6-1) + (6-0)(6-3)$$
$$+ (6-1)(6-3)] \times 1$$

$$= -42 - 44 + (30+18+15) = -23$$

Q: find the first derivative of $y = 2e^x - x - 1$, tabulated below at the point $x = 0.65$:

x	y
0.4	1.5836494
0.5	1.7974426
0.6	2.0442376
0.7	2.3275054
0.8	2.6510180

Solution: To find $y'(0.65)$ of the function $y = 2e^x - x - 1$ from the given table, we use Gauss's backward interpolation formula

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \cdots$$

when $u = \dfrac{x - x_0}{h}$

$$y'(x) = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_{-1} + \frac{1}{2}(2u+1)\Delta^2 y_{-1} + \frac{1}{6}(3u^2-1)\Delta^3 y_{-2} + \frac{1}{24}(4u^3 + 6u^2 - 2u - 2)\Delta^4 y_{-2} + \cdots\right]$$

difference table

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	0.4	1.5836494				
			0.2137932			
-1	0.5	1.7974426		0.0330018		
			0.2467950		0.003471	
0	0.6	2.0442376		0.0364728		0.000301
			0.2832678		0.003772	
1	0.7	2.3275054		0.0402448		
			0.3235126			
2	0.8	2.6510180				

$\because x = 0.65,\ x_0 = 0.6,\ h = 0.1$

$\therefore u = \dfrac{0.65 - 0.6}{0.1} = 0.5$

$$y'(0.65) = \frac{1}{0.1}\Big[0.2467950 + \frac{1}{2}(2\times 0.5+1)\times(0.0364728)$$
$$+\frac{1}{6}\left(3\times(0.5)^2-1\right)\times 0.003471$$
$$+\frac{1}{24}\left\{4\times(0.5)^3+6\times(0.5)^2-2\times(0.5)-2\right\}\times 0.000301\Big]$$

$$= 10\Big[0.2467950 + 0.0364728 + \frac{1}{6}\times(-0.25)\times 0.003471$$
$$+\frac{1}{24}(0.5+1.5-1-2)\times(0.000301)\Big]$$

$$= 10\left[0.2467950 + 0.0364728 - 0.0001446 - 0.0000125\right]$$

$$y'(0.65) = 2.831107 \simeq 2.8311$$

verification

$$y = 2e^x - x - 1$$

$$y' = 2e^x - 1$$

$$y'(0.65) = 2\times e^{(0.65)} - 1$$

$$= 2\times 1.915540829 - 1$$

$$= 2.831081658$$

$$\simeq 2.8311$$

प्रश्नावली (Exercise) VI

1. निम्न सारणी के फलन का $x=3.0$ पर प्रथम तथा द्वितीय अवकलन ज्ञात कीजिये।
Find the first and second derivatives of the function tabulated below at the point $x=3.0$.

x :	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$:	−14.000	−10.032	−5.296	0.256	6.672	14.000

[Ajmer 04]

2. निम्न सारणी से केन्द्रीय अन्तर बनाकर $x=1$ पर $\frac{dy}{dx}$ ज्ञात कीजिये:

Find $\frac{dy}{dx}$ at $x=1$ from the following table by constructing a central difference table :

x :	1	2	3	4	5	6
y :	198669	295520	389418	479425	564642	644217

[Raj. 12]

3. निम्न सारणी से $\frac{dy}{dx}$ तथा $\frac{d^2y}{dx^2}$ का मान $x=500$ पर ज्ञात कीजिए:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=500$ from the following table :

x :	500	510	520	530	540	550
y :	6.214608	6.234411	6.253829	6.272877	6.291569	6.309918

4. निम्न सारणिक फलन की $x=1.5$ पर प्रथम, द्वितीय तथा तृतीय अवकलज ज्ञात कीजिये:

Find the first second and third derivatives of the function tabulated below at the point $x=1.5$:

x :	1.5	2.00	2.5	3.00	3.5	4.00
$y=f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

[Bikaner 17; Raj. 12]

5. निम्न आंकड़ों का प्रयोग करते हुए फलन $f(x)=\dfrac{1}{1+x^2}$ के लिये $f'(1)$ का मान ज्ञात कीजिए:

Find $f'(1)$ for $f(x)=\dfrac{1}{1+x^2}$ usin the following data:

x	:	1.0	1.1	1.2	1.3	1.4
$f(x)$	:	0.25	0.2268	0.2066	0.1890	0.1736

6. $y=x^{1/3}$ का $x=50$ पर $\dfrac{dy}{dx}$ तथा $\dfrac{d^2y}{dx^2}$ के मान निम्न सारणी से ज्ञात कीजिए:

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ of $y=x^{1/3}$ at $x=50$ from the following table :

x	:	50	51	52	53	54
y	:	3.6840	3.7084	3.7325	3.7563	3.7798
x	:	55	56			
y	:	3.8030	3.8259			

[Jodhpur [illegible]]

7. निम्न सारणी $f(x)$ का $x=0.4$ पर प्रथम अवकलज ज्ञात कीजिये:

Find the first derivative of $f(x)$ at $x=0.4$ from the following table :

x	:	0.1	0.2	0.3	0.4
$f(x)$	:	1.10517	1.22140	1.34986	1.49182

[Sikar 18; Alwar 17; Jodhpur 03, 18; Ajmer 13]

8. न्यूटन–ग्रेगरी पश्च अन्तर्वेशन सूत्र से $(dy/dx)_{x=x_0}$ के मान की गणना करने हेतु सूत्र स्थापित कीजिये। निम्न से $x=6$ पर $\dfrac{dy}{dx}$ तथा $\dfrac{d^2y}{dx^2}$ का मान ज्ञात कीजिये:

Establish the formula for computing the value of $(dy/dx)_{x=x_0}$ from Newton's-Gregory backward interpolation formula. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

at $x = 6$ from the following table:

x:	0	1	2	3	4	5	6
$f(x)$:	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

9. फलन $y = \sin x$ के मान निम्न सारणी में दिये हुए हैं, $x = 1$ पर अवकलज का मान ज्ञात कीजिये।

The function $y = \sin x$ is tabulated in the scheme below. Find the derivative at the point $x = 1$.

x:	0.7	0.8	0.9	1.0	1.1
y:	0.644218	0.717356	0.783327	0.841471	0.891207

x:	1.2	1.3
y:	0.932039	0.963558

10. निम्न सारणी द्वारा स्टर्लिग सूत्र का प्रयोग कर फलन $y = 2e^x - x - 1$ का $x =$ 0.6 पर प्रथम अवकलज ज्ञात कीजिये।

Use stirling's formula to find the first derivative of the function $y = 2e^x - x - 1$ tabulated below at the point $x = 0.6$.

x	y
0.4	1.5836494
0.5	1.7974426
0.6	2.0442376
0.7	2.3275054
0.8	2.6510818

इसकी तुलना शुद्ध मान से करिये जो कि 2.044238 है।

Compare with the true value which is 2.044238. [Jodhput 04, 05]

11. निम्न सारणी से प्रथम दो अवकलज, फलन $f(x)$ के $x = 1$ पर ज्ञात कीजिये।

Find the first two derivatives fo $f(x)$ at $x = 1$ from the following table:

x:	−2	−1	0	1	2	3	4
$f(x)$:	104	17	0	−1	8	69	272

[Raj. 07]

12 किसी मशीन में एक स्लाईडर एक स्थिर छड़ पर घूमता है। स्लाईडर की छड़ पर दूरी x से.मी. भिन्न–भिन्न समय t से. पर निम्नलिखित सारणी में दी गई है।

$t=0.3$ से. पर स्लाईडर का वेग और त्वरण ज्ञात कीजिए।

A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values for the time t second. Find the velocity and acceleration of the slider when $t=0.3$ second.

t:	0	0.1	0.2	0.3	0.4	0.5	0.6
x:	30.13	31.62	32.87	33.64	33.95	33.81	33.24

13. निम्नांकित युग्ल x तथा $f(x)$ के मानों से $f'(4)$ का मान ज्ञात कीजिये:

Given the following pairs of values of x and $f(x)$ find $f'(4)$:

x :	1	2	4	8	10
$f(x)$:	0	1	5	21	25

14. निम्न आंकड़ों से $f'(10)$ ज्ञात कीजिए। $f'(18)$ का मान भी ज्ञात कीजिए।

Find $f'(10)$ from the floowing data :

x :	3	5	11	27	34
$f(x)$:	−13	23	889	17315	35606

[Alwar 18]

Find also $f'(18)$. **[Raj. 13, 17; Jodhpur 03; Ajmer 05]**

15. निम्न सारणी से $f'(6)$ ज्ञात कीजिये:

Find $f'(6)$ from the following table :

x :	0	1	3	4	5	7	9
$f(x)$:	150	108	0	−54	−100	−144	−84

16. निम्न सारणिक फलन की $x=2.5$ पर प्रथम, द्वितीय तथा तृतीय अवकलज ज्ञात कीजिये :

Find the first, second and third derivatives of the function tabulated below at the pint $x=2.5$:

x :	1.5	1.9	2.5	3.2	4.3	5.9
$y=f(x)$:	3.375	6.059	13.625	22.368	73.901	196.579

17. निम्न सारणी से $f'''(5)$ ज्ञात कीजिये:

x :	2	4	9	13	16	21	29
$f(x)$:	57	1345	66340	402052	1118209	4287844	21242820

18. आधार 10 पर $x=300$ से 310 तक इकाई वृद्धि देते हुए दस दशमलव तक लघुगुणक लेकर $\log_{10}x$ का अवकलज $x=310$ पर ज्ञात कीजिये।

Take 10 figure logarithm to base 10 from $x = 300$ to $x = 310$ by unit increments in x. Calculate the first derivative of $\log_{10} x$, when $x = 310$.

19. घात छः तक स्टर्लिंग सूत्र का प्रयोग कर सिद्ध कीजिये।

By using stirling's of degree six, prove that

(a) $y_0' = \frac{1}{h}\left[\delta\mu y_0 - \frac{1}{6}\delta^3\mu y_0 + \frac{1}{30}\delta^5\mu y_0 + \dots\right]$

(b) $y_0'' = \frac{1}{h^2}\left[\delta^2 y_0 - \frac{1}{12}\delta^4 y_0 + \frac{1}{90}\delta^6 y_0 - \dots\right]$ [Kota 04, 08]

20. बेसल सूत्र से निम्न सन्निकटन का निगमन कीजियेः

Deduce the following approximations from the Bessel's formula:

(i) $\frac{d}{dx}(y_x) = \Delta y_{x-(1/2)} - \frac{1}{24}\Delta^3 y_{x-(3/2)}$

(ii) $\frac{d^2}{dx^2}(y_x) = \frac{1}{2}[\Delta^2 y_{x-(3/2)} + \Delta^2 y_{x-(1/2)}]$

(iii) $\frac{d^3}{dx^3}(y_x) = \Delta^3 y_{x-(3/2)}$.

21. सिद्ध कीजिए कि (Prove that) :

(i) $y' = \frac{1}{h}\left[\delta h - \frac{\delta^3 y}{24} + \frac{3}{640}\delta^5 y - \dots\right]$ [Bikaner 18; Jodhpur 06; Udipur 07, 08

(ii) $y'' = \frac{1}{h^2}\left[\delta^2 y - \frac{1}{12}\delta^4 y + \frac{1}{90}\delta^6 y\right]$

22. दिया गया है (Given that) :

x	:	1.0	1.1	1.2	1.3	1.4	1.5
y	:	7.98	8.40	8.78	9.13	9.45	9.75

[Kota 1

ज्ञात कीजिये (Find) : $\frac{dy}{dx}$.

प्रश्नावली (Exercise) VI

1. 18 ; 18 2. 98008
3. 0.00200 4. 4.750 ; 9.000 ; 6.000
5. –0.5
6. $\left(\frac{dy}{dx}\right)_{x=50} = 0.2455;\ \left(\frac{d^2y}{dx^2}\right)_{x=50} = -0.0003$
7. 1.49139, 0.54030 10. 2.644225 11. 1;6
12. $v = 5.34$ cm/sec.; $f = -45.6$ c/sec^2
13. 2.8333 14. 233
15. –23
16. $f'(2.5) = 16.750,\ f''(2.5) = 15,\ f'''(2.5) = 6$
17. 1626 18. 0.0018

Q: By using stirling's of degree six, prove that

(i) $y_0' = \frac{1}{h}\left[\delta\mu y_0 - \frac{1}{6}\delta^3\mu y_0 + \frac{1}{30}\delta^5\mu y_0 + \cdots\right]$

(ii) $y_0'' = \frac{1}{h^2}\left[\delta^2 y_0 - \frac{1}{12}\delta^4 y_0 + \frac{1}{90}\delta^6 y_0 - \cdots\right]$

Solution: striling's formula

$$y_x = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$
$$+ \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right]$$
$$+ \frac{u^2(u^2-1)(u^2-4)}{6!}\Delta^6 y_{-3} + \cdots \quad \text{--- (1)}$$

where $u = \frac{x - x_0}{h}$ --- (2)

let $x_0 = 0,\ h = 1 \Rightarrow u = x$

$$y_x = y_0 + x\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{x^2}{2!}\Delta^2 y_{-1} + \frac{x(x^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$
$$+ \frac{x^2(x^2-1)}{4!}\Delta^4 y_{-2} + \frac{x(x^2-1)(x^2-4)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right]$$
$$+ \frac{x^2(x^2-1)(x^2-4)}{6!}\Delta^6 y_{-3} + \cdots \quad \text{--- (3)}$$

Differentiate equation (3), w.r.t x, we get

$$\frac{d}{dx}(y_x) = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + x\Delta^2 y_{-1} + \frac{(3x^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{(4x^3-2x)}{4!}\Delta^4 y_{-2} + \frac{(5x^4-15x^2+4)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)$$
$$+ \frac{(6x^5-20x^3+8x)}{6!}\Delta^6 y_{-3} + \cdots \quad \text{--- (4)}$$

x shifted at origin

ie put $x = 0$ in eqn (4), we get

$$\frac{d}{du}(y_0) = \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots$$

$[\because u = x]$

$$= \frac{1}{2}(\Delta y_0 + \Delta E^{-1} y_0) - \frac{1}{12}(\Delta^3 E^{-1} y_0 + \Delta^3 E^{-2} y_0) + \frac{1}{60}(\Delta^5 E^{-2} y_0 + \Delta^5 E^{-3} y_0) + \cdots$$

$$= \frac{1}{2}(\Delta + \Delta E^{-1}) y_0 - \frac{1}{12}(\Delta^3 E^{-1} + \Delta^3 E^{-2}) y_0 + \frac{1}{60}(\Delta^5 E^{-2} + \Delta^5 E^{-3}) y_0 + \cdots$$

$$= \frac{1}{2}(\Delta + \Delta E^{-1}) y_0 - \frac{1}{12}\Delta^2 E^{-1}(\Delta + \Delta E^{-1}) y_0 + \frac{1}{60}\Delta^4 E^{-2}(\Delta + \Delta E^{-1}) y_0 + \cdots$$

$$= \mu\delta y_0 - \frac{1}{6}\delta^2(\mu\delta) y_0 + \frac{1}{30}\delta^4(\mu\delta) y_0 + \cdots \quad (5)$$

$\because \mu\delta \equiv \frac{1}{2}(\Delta + \Delta E^{-1})$ & $\delta \equiv \Delta E^{-1/2}$

by eq^n(1) & (2)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

eq^n(5) becomes

$$\frac{d}{dx}(y_0) = \frac{1}{h}\left(\delta\mu y_0 - \frac{1}{6}\delta^3 \mu y_0 + \frac{1}{30}\delta^5 \mu y_0 + \cdots\right)$$

Again differentiate eq^n(4), w.r.t x we get

$$y''_x = \Delta^2 y_{-1} + x\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{(12x^2 - 2)}{4!}\Delta^4 y_{-2} + \frac{(20x^3 - 30x)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \frac{(30x^4 - 60x^2 + 8)}{6!}\Delta^6 y_{-3} +$$

put $x = 0$, we get

$$\frac{d^2}{dx^2}(y_0) = \Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \cdots$$

$$\frac{d^2}{du^2}(y_0) = \Delta^2 E^{-1} y_0 - \frac{1}{12}\Delta^4 E^{-2} y_0 + \frac{1}{90}\Delta^6 E^{-3} y_0 + \cdots$$

$\because$ $y_0'' = \frac{d^2}{dx^2}(y_0) = \frac{1}{h^2}\frac{d^2}{du^2}(y_0)$ & $\Delta E^{-1} \equiv \delta$

$$\therefore y_0'' = \frac{1}{h^2}\left(\delta^2 y_0 - \frac{1}{12}\delta^4 y_0 + \frac{1}{90}\delta^6 y_0 - \cdots\right)$$

Unit - II Numerical Integration

Quadrature formulae:

let $I = \int_a^b y\,dx$, where $y = f(x)$ — (1)

be a finite integral, also let we have values of $f(x)$ at equidistant values of x ie $x_0, x_0+h, x_0+2h, \ldots\ldots x_0+nh$

therefore corresponding value of y are $y_0, y_1, y_2 \ldots\ldots, y_n$

then

$$I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y_x\,dx \quad — (2)$$

let $u = \dfrac{x-x_0}{h} \Rightarrow du = \dfrac{dx}{h}$

then (2) $\Rightarrow$

$$I = \int_0^n (y_{x_0+hu})\,h\,du$$

$$= h\int_0^n (y_{x_0+hu})\,du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \right] du$$

$$\int_{x_0}^{x_n} y_x\,dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots \text{upto } (n+1) \text{ term}\right]$$

when $x_n = x_0 + nh$

is called quadrature formula

by putting the values of $n = 1, 2, 3 \ldots$ we get different formula's

Trapezoidal Rule: $I = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+ \text{---} + y_{n-1})\right]$

proof: put $n=1$ in quadrature formula we get

$$\int_{x_0}^{x_1} y_x\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}\left[2y_0 + y_1 - y_0\right]$$

$$\int_{x_0}^{x_1} y_x\,dx = \frac{h}{2}\left[y_0+y_1\right] \quad \text{---} \quad (1)$$

similary

$$\int_{x_1}^{x_2} y_x\,dx = \frac{h}{2}\left[y_1+y_2\right]$$

$$\int_{x_2}^{x_3} y_x\,dx = \frac{h}{2}\left[y_2+y_3\right]$$

$$\vdots$$

$$\int_{x_{n-1}}^{x_n} y_x\,dx = \frac{h}{2}\left[y_{n-1}+y_n\right]$$

$$\therefore \int_{x_0}^{x_n} y_x\,dx = \int_{x_0}^{x_1} y_x\,dx + \int_{x_1}^{x_2} y_x\,dx + \int_{x_2}^{x_3} y_x\,dx + \text{---} + \int_{x_{n-1}}^{x_n} y_x\,dx$$

$$\therefore I = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+ \text{---} + y_{n-1})\right]$$

Simpson's $\left(\frac{1}{3}\right)$ Rule: $I = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+ \cdots + y_{n-2}) + 4(y_1+y_3+ \cdots + y_{n-1})\right]$

proof:

put $n=2$ in quadrature formula, we get

$$\int_{x_0}^{x_2} y_x dx = h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3}-2\right)\frac{\Delta^2 y_0}{2!}\right]$$

$$= h\left[2y_0 + 2(y_1-y_0) + \frac{1}{3}(y_2-2y_1+y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly $\int_{x_2}^{x_4} y_x dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$

$$\int_{x_4}^{x_6} y_x dx = \frac{h}{3}\left[y_4 + 4y_5 + y_6\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\int_{x_{n-2}}^{x_n} y_x dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

$$\therefore \int_{x_0}^{x_n} y_x dx = \int_{x_0}^{x_2} y_x dx + \int_{x_2}^{x_4} y_x dx + \cdots + \int_{x_{n-2}}^{x_n} y_x dx$$

$$\therefore I = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+ \cdots + y_{n-2}) + 4(y_1+y_3+ \cdots + y_{n-1})\right]$$

Simpson's $\left(\frac{3}{8}\right)$ Rule: $I = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+\dots+y_{n-3}) + 3(y_1+y_2+y_4+\dots+y_{n-1})\right]$

Q: find the value of $\int_0^1 \frac{1}{(1+x)}\,dx$ by using integration Rule's and compaire with exact answere

Sol: let $I = \int_0^1 \frac{1}{1+x}\,dx = \int_a^b y\,dx$

where $y = \frac{1}{1+x}$, $a=0$, $b=1$

To use Numerical Integration Rule's divide interval $[0,1]$ in 6 equal subparts (subintervals)

i.e. $h = \frac{1}{6}$

then value of x are $x_0 = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 = x_6$

and corresponding value of y

ie

x:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
y:	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By trapezoidal Rule:

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{6}{7}+\frac{3}{4}+\frac{2}{3}+\frac{3}{5}+\frac{6}{11}\right)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 2(0.8571+0.75+0.6667+0.6+0.5455)\right]$$

$$= \frac{1}{12}\left[1.5 + 6.8386\right]$$

$$\boxed{I \cong 0.6949}$$

(ii) By simpson's $(\frac{1}{3})$ Rule.

$$I = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{18}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{3}{4}+\frac{3}{5}\right) + 4\left(\frac{6}{7}+\frac{2}{3}+\frac{6}{11}\right)\right]$$

$$\boxed{I \simeq 0.6931}$$

(iii) By simpson's $(\frac{3}{8})$ Rule:

$$I = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{1}{16}\left[\left(1+\frac{1}{2}\right) + 2\times\frac{2}{3} + 3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right)\right]$$

$$\boxed{I \simeq 0.6932}$$

we get

$$I = \int_0^1 \frac{1}{1+x}\,dx \simeq 0.6949 \text{ (Trapezoidal)}$$
$$\simeq 0.6931 \text{ (simp. } \tfrac{1}{3})$$
$$\simeq 0.6932 \left[\text{simpson's } (\tfrac{1}{3}) \ \& \ (\tfrac{3}{8})\right]$$

and

$$\int_0^1 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^1$$

$$= \log 2 - \log 1$$

$$= 0.6931 - 0$$

$$= 0.6931$$

Hence by simpson's $(\frac{1}{3})$ Rule we get approximate correct answer

error : Exact value − Approximate value

(i) By Trapezoidal : $0.6931 - 0.6949 \simeq -0.0018$

(ii) By Simpson's $(\frac{1}{3})$: $0.6931 - 0.6931 = 0$

(iii) By simpson's $(\frac{3}{8})$: $0.6931 - 0.6932 \simeq 0.0001$

Q: Use simpson's '$\frac{1}{3}$' Rule to evaluate $\int_0^{16} y\,dx$ from the following data:

x:	0	2	4	6	8	10	12	14	16
y:	0	4	7	9	12	15	14	8	3

Sol: Given data:

x:	0	2	4	6	8	10	12	14	16
y:	0	4	7	9	12	15	14	8	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

To evaluate $\int_0^{16} y\,dx$, simpson's '$\frac{1}{3}$' rule

$$\int_0^{16} y\,dx = \frac{h}{3}\left[(y_0+y_8)+2(y_2+y_4+y_6)+4(y_1+y_3+y_5+y_7)\right]$$

where $h=2$

$$= \frac{2}{3}\left[(0+3)+2(7+12+14)+4(4+9+15+8)\right]$$

$$= \frac{2}{3}\left[3+66+144\right]$$

$$= \frac{2}{3}\times 213$$

$$\Rightarrow \int_0^{16} y\,dx = 142$$

Q: Calculate by simpson's '$\frac{1}{3}$' Rule an approximate value of $\int_{-3}^{3} x^4\,dx$ by taking seven equidistant ordinates, compare it with exact value and the value obtained by using trapezoidal rule. comment its comparison.

Sol: let $I = \int_{-3}^{3} x^4\,dx = \int_a^b y\,dx$

where $a=-3,\ b=3,\ y=x^4$

divide the interval $[-3,3]$ in seven equidistant ordinates

i.e. $h = \frac{3-(-3)}{6} = 1$

therefore

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x:	−3	−2	−1	0	1	2	3
y:	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's '$\frac{1}{3}$' Rule

$$I = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left[(81+81) + 2(1+1) + 4(16+0+16)\right]$$

$$= \frac{1}{3}\left[162 + 4 + 128\right] = 98$$

Trapezoidal Rule

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(81+81) + 2(16+1+0+1+16)\right]$$

$$= \cancel{\frac{1}{2}[162+} \; 81 + 34 = 115$$

exact value $\quad I = \int_{-3}^{3} x^4\,dx = 2\int_{0}^{3} x^4\,dx$

$$= 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2}{5}\times 3^5 = \frac{2}{5}\times 243 = 97.2$$

Error: Exact value – Approximate value

by Simpson's $\frac{1}{3}$: $\quad 97.2 - 98 = -0.8$

by trapezoidal: $\quad 97.2 - 115 = -17.8$

Hence Simpson's '$\frac{1}{3}$' Rule give more approximate value of the integral $\int_{-3}^{3} x^4\,dx$.

प्रश्नावली (Exercise) VII

1. ट्रेपीजोइडल नियम द्वारा निम्न का मान ज्ञात कीजिये:
Using Trapezoidal rule calculate the value of

$$\int_{0.2}^{1.4} e^x \, dx$$ or by Simpson $\frac{1}{3}$ rule (अथवा सिम्पसन $\frac{1}{3}$ नियम)

जबकि $h = 0.1$ इस मान की यथार्थ मान से तुलना कीजिये।

Given $h = 0.1$ compare it with the exact value.

[Ajmer 09; Jodhpur 02; Kota 13; Jodpur 16]

5. निम्न आंकड़ो से $\int_0^{16} y\,dx$ का मान सिम्पसन के $\frac{1}{3}$ नियम द्वारा ज्ञात कीजिए:

Use Simpson's '$\frac{1}{3}$' rule to evaluate $\int_0^{16} y\,dx$ from the following data:

x:	0	2	4	6	8	10	12	14	16
y:	0	4	7	9	12	15	14	8	3

[Udaipur 06; Bikaner 06]

6. सिम्पसन के '$\frac{1}{3}$' तथा '$\frac{3}{8}$' नियमों के द्वारा निम्न समाकलन का मान ज्ञात कीजिए:

Use Simpson's '$\frac{1}{3}$' and '$\frac{3}{8}$' rule to evaluate:

$$\int_0^1 \frac{dx}{1+x}$$

[Bikaner 08, 12, Raj. 07]

दोनों स्थितियों में प्राप्त $\log_e 2$ के मानों की तुलना कीजिये।

Compare the values of $\log_e 2$ in the two cases.

7. **[Bikaner 05; ...**

प्रश्नावली (Exercise) VII

1. 2.613604
2. 0.9981
3. 1.136
4. 0.692325
6. 0.69315
7. 98 ; सही मान = 72.2 ट्रेपिजोयडल नियम द्वारा = 115
8. 710 वर्ग मीटर
9. 1.2996
10. 3.150723
11. (a) 0.9981 (b) 1.006

Newton - Cote Quadrature formula:

let $I = \int_a^b f(x)\,dx = \int_a^b y\,dx$, $y = f(x)$

also for $(n+1)$ equidistant values of x

i.e. $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, - - -, $x_n = x_0 + nh = b$

the corresponding values of $f(x)$ are

$y_0, y_1, y_2, \text{- - -}, y_n$

then $I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} y_x\,dx$

$\because u = \dfrac{x - x_0}{h}$

$\therefore x = x_0 + uh$

$dx = h\,du$

$\Rightarrow I = h \int_0^n y_u\,du$

by Newton - Gregory forward interpolation formula

$$I = h\int_0^n \left[y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \text{---} \right.$$
$$\left. \text{---} + \frac{u^{(n)}}{n!}\Delta^n y_0 \right] du$$

where $u^{(n)} = u(u-1)(u-2) \text{---} [u-(n-1)]$

$$I = h\left[a_0 y_0 + a_1 \Delta y_0 + a_2 \Delta^2 y_0 + \text{---} + a_n \Delta^n y_0 \right]$$
$$= h \sum_{i=0}^{n} a_i \Delta^n y_0 \quad \text{---} (1)$$

when $a_i = \dfrac{1}{i!} \int_0^n u^{(i)}\,du$

Called Newton - Cote Quadrature formula.

Now $a_0 = \frac{1}{0!}\int_0^n u^{(0)}\,du = n$

$$a_1 = \frac{1}{1!}\int_0^n u^{(1)}\,du = \int_0^n u\,du = \frac{n^2}{2}$$

$$a_2 = \frac{1}{2!}\int_0^n u^{(2)}\,du = \frac{1}{2}\int_0^n u(u-1)\,du$$

$$= \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)$$

Similar $a_3 = \frac{1}{3!}\int_0^n u^{(3)}\,du = \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)$

$$a_4 = \frac{1}{24}\left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)$$

$$a_5 = \frac{1}{120}\left(\frac{n^6}{6} - 2n^5 + \frac{35}{4}n^4 - \frac{50}{3}n^3 + 12n^2\right)$$

$\vdots \qquad \vdots$

and $\Delta y_0 = y_1 - y_0$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

$\vdots \qquad \vdots$

then $\cancel{I = h[a_0 y_0 + \frac{n^2}{2}(y_1 - y_0)}$

$$I = h[a_0 y_0 + a_1(y_1 - y_0) + a_2(y_2 - 2y_1 + y_0)$$

$$+ a_3(y_3 - 3y_2 + 3y_1 - y_0) + a_4(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)$$

$$\cancel{+ - -]} + a_5(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0) + -]$$

$$= h\Big[(a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots) y_0$$

$$+ (a_1 - 2a_2 + 3a_3 - 4a_4 + \cdots) y_1$$

$$+ (a_2 - 3a_3 + 6a_4 - 10a_5 + \cdots) y_2$$

$$+ (a_3 - 4a_4 + 10a_5 - 20a_6 + \cdots) y_3$$

$$+ \cdots \Big]$$

$$= h\Big[(a_0 - {}^1c_1 a_1 + {}^2c_2 a_2 - {}^3c_3 a_3 + {}^4c_4 a_4 - {}^5c_5 a_5 + \cdots) y_0$$

$$+ (a_1 - {}^2c_1 a_2 + {}^3c_1 a_3 - {}^4c_1 a_4 + \cdots) y_1$$

$$+ (a_2 - {}^3c_2 a_3 + {}^4c_2 a_4 - {}^5c_2 a_5 + \cdots) y_2$$

$$+ (a_3 - {}^4c_3 a_4 + {}^5c_3 a_5 - {}^6c_3 a_6 + \cdots) y_3$$

$$+ \cdots \Big]$$

$\boxed{\cancel{I = h \sum_{i=0}^{n} \lambda_i y_i}}$ $\quad I = h \sum_{i=0}^{n} \lambda_i y_i$

where $\lambda_0 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$

$\lambda_1 = a_1 - 2a_2 + 3a_3 - 4a_4 + \cdots$

$\lambda_2 = a_2 - 3a_3 + 6a_4 - 10a_5 + \cdots$

$\lambda_i = i(0, 1, 2 \ldots, n)$ are called Cotes Numbers

Newton - Cotes quadrature formula (By Lagrange's method)

Let $I = \int_a^b y\,dx$, $y = f(x)$

also for $n+1$ values of x at equal interval

ie $x_0, x_1, \dots x_n$, when $x_0 = a$, $x_n = b = x_0 + nh$

Corresponding values of $f(x)$ are

$f(x_0), f(x_1), \dots f(x_n)$

then $I = \int_{x_0}^{x_0+nh} f(x)\,dx$

$\because$ by Lagrange's formula

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1)$$

$$+ \dots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})} f(x_n)$$

$$\therefore I = h \sum_{i=0}^{n} \alpha_i f(x_i)$$

Called Newton Cotes quadrature formula

where $$\alpha_i = \frac{1}{h(x_i-x_0)(x_i-x_1)\dots(x_i-x_n)} \int_{x_0}^{x_0+nh} \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(x-x_i)}\,dx$$

if $u = \frac{x-x_0}{h}$

then $$\alpha_i = \frac{(-1)^{n-i}}{i!\,(n-i)!} \int_0^n \frac{u(u-1)(u-2)\dots(u-n)}{(u-i)}\,du$$

$\alpha_i\ (i = 0, 1, 2 \dots n)$ called Newton Cotes numbers

Theorem: The cotes Numbers are symmetrical from both ends:

Pf: Newton cotes numbers are given by

$$d_i = \frac{(-1)^{n-i}}{i!\,(n-i)!}\int_0^n \frac{u(u-1)(u-2)\cdots(u-n+1)(u-n)}{(u-i)}\,du$$

Now $i \to n-i$ & put $n-u = s$, then

$$d_{n-i} = \frac{(-1)^{i}}{(n-i)!\,i!}\int_n^0 \frac{(n-s)(n-s-1)(n-s-2)\cdots(-s+1)(-s)}{[(n-s)-(n-i)]}(-ds)$$

$$= \frac{(-1)^{i}}{(n-i)!\,i!}\int_n^0 (-1)^{n+2}\frac{(s-n)(s-n+1)\cdots(s-1)s}{(-s+i)}\,ds$$

$$= \frac{(-1)^{n-i}}{i!\,(n-i)!}\int_0^n \frac{(s-n)(s-n+1)\cdots(s-1)s}{(s-i)}\,ds$$

$$\boxed{d_{n-i} = d_i}$$

Theorem: The sum of cotes numbers is equal to the number of subdivisions of the range of integration:

ie $\sum_{i=0}^{n} d_i = n$

Pf: let $f(x) = f(x_i) = 1$, $\forall\ i = 0,1,2 \cdots n$

then $I = \int_{x_0}^{x_0+nh} f(x)\,dx = h\sum_{i=0}^{n} d_i f(x_i)$ by Newton cotes for

$$\Rightarrow \int_{x_0}^{x_0+nh} dx = h\sum_{i=0}^{n} d_i$$

$$\Rightarrow nh = h\sum_{i=0}^{n} d_i$$

$$\Rightarrow \boxed{\sum_{i=0}^{n} d_i = n}$$

Q: Deduct Trapezoidal, Simpson's Rule by Cote's formula

Sol Newton Cote's formula

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = h \sum_{i=0}^{n} d_i f(x_i) \quad \text{---} (1)$$

when $f(x) = y$, when $d_i = \frac{(-1)^{n-i}}{i!(n-i)!} \int_0^n \frac{u(u-1)--(u-n)}{(u-i)} du$

(i) Trapezoidal Rule

put $n=1$ in (1), we get

$$\int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_1} y\,dx = h\left[d_0 f(x_0) + d_1 f(x_1)\right] \quad \text{---} (2)$$

$$\text{when } d_0 = \frac{(-1)^{1-0}}{0!(1-0)!} \int_0^1 \frac{u(u-1)}{(u-0)} du$$

$$= \frac{-1}{1\times 1} \int_0^1 (u-1)\,du$$

$$= \frac{1}{2}$$

$$\& \ d_1 = \frac{(-1)^0}{1!\,0!} \int_0^1 \frac{u(u-1)}{(u-1)} du$$

$$= \int_0^1 u\,du = \frac{1}{2}$$

then (2) $\Rightarrow$

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)$$

which is same as by ~~qua~~ general quadrature formula

Hence $I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \text{---} + y_{n-1})\right]$

(ii) Simpson's '$\frac{1}{3}$' Rule : Put $n = 2$ in (1), we get

$$\int_{x_0}^{x_0+2h} y\,dx = \int_{x_0}^{x_2} y\,dx = h\left[\lambda_0 f(x_0) + d_1 f(x_1) + d_2 f(x_2)\right] \quad \text{---} (3)$$

$$\text{Where } d_0 = \frac{(-1)^2}{0!\,2!} \int_0^2 \frac{u(u-1)(u-2)}{u} du$$

$$= \frac{1}{2}\int_0^2 (u^2-3u+2)\,du = \frac{1}{2}\left[\frac{u^3}{3} - \frac{3u^2}{2} + 2u\right]_0^2 = \frac{1}{2}\left(\frac{8}{3} - 6 + 4\right)$$

$$\lambda_0 = \frac{1}{3}$$

$$\lambda_1 = \frac{(-1)^1}{1!\,1!}\int_0^2 \frac{u(u-1)(u-2)}{(u-1)}\,du = \frac{4}{3}$$

$$\lambda_2 = \frac{1}{3}$$

thus (3) $\Rightarrow$ $\int_{x_0}^{x_2} y\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$

which is same as by General quadrature formula

Hence $I = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\right]$

Similarly Simpson's $\left(\frac{3}{8}\right)$ Rule