Unit - III

Gauss's quadrature formula :

$$\int_{-1}^{1} F(u)\,du = \sum_{i=0}^{n} w_i F(u_i)$$

where w_i & u_i are called weights & Nodes

and get from $(2n+2)$ nonlinear algebraic equations.

Case (i) : $n=0$ (one point formula)

$$\int_{-1}^{1} F(u)\,du = w_0 F(u_0) = 2F(0)$$

$$\& \int_{a}^{b} f(x)\,dx = \frac{(b-a)}{2}\int_{-1}^{1} F(u)\,du$$

$$= (b-a) f\left(\frac{a+b}{2}\right)$$

Rate of convergence is one

Case (ii) : $n=1$ (Two point formula)

$$\int_{-1}^{1} F(u)\,du = w_0 F(u_0) + w_1 F(u_1)$$

$$= 1.F\left(-\frac{1}{\sqrt{3}}\right) \cancel{+ 1.F\left(\frac{1}{\sqrt{3}}\right)} + 1.F\left(\frac{1}{\sqrt{3}}\right)$$

$$\& \int_{a}^{b} f(x)\,dx = \frac{(b-a)}{2}\int_{-1}^{1} F(u)\,du = \frac{(b-a)}{2}\left[F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

Rate of convergence is Three

Case (iii) $n = 2$ (~~Two~~ Three point formula)

$$\int_{-1}^{1} F(u)\,du = w_0 F(u_0) + w_1 F(u_1) + w_2 F(u_2)$$

$$= \frac{5}{9} F\left[-\sqrt{\frac{3}{5}}\right] + \frac{8}{9} F(0) + \frac{5}{9} F\left[\sqrt{\frac{3}{5}}\right]$$

& $\int_a^b f(x)\,dx = \frac{(b-a)}{2} \int_{-1}^{1} F(u)\,du$

Rate of converges is 5

Conversion of integral $\int_a^b f(x)\,dx$ to $\int_{-1}^{1} F(u)\,du$

let $x = \frac{(b-a)}{2} u + \frac{(a+b)}{2}$

$dx = \frac{(b-a)}{2} du$

then $\int_a^b f(x)\,dx = \frac{(b-a)}{2} \int_{-1}^{1} F(u)\,du$

where $F(u) = f\left[\frac{(b-a)}{2} u + \frac{(a+b)}{2}\right]$

Q: Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using Gauss quadrature formula for three point.

Sol. let $I = \int_0^1 \frac{1}{1+x^2}\,dx = \int_a^b f(x)\,dx$

$\Rightarrow [a,b] = [0,1]$, $f(x) = \frac{1}{1+x^2}$

let $x = \frac{(b-a)}{2} u + \frac{(a+b)}{2} \Rightarrow x = \frac{u}{2} + \frac{1}{2}$

$\Rightarrow dx = \frac{1}{2}$

$\Rightarrow u = 2x - 1,$ ~~f(x) =~~

~~then I = $\int_0^1 \frac{1}{1+x^2}\,dx$~~ =

& $f(x) = \frac{1}{1+x^2} = \frac{1}{1 + \left[\frac{u+1}{2}\right]^2}$

$$= \frac{4}{u^2+2u+5}$$

then $I = \int_0^1 \frac{1}{1+x^2}\,dx = 2\int_{-1}^{1} \frac{1}{(u^2+2u+5)}\,du$

$$= 2\int_{-1}^{1} F(u)\,du$$

by Gauss quadrature formula for three point

$$\int_{-1}^{1} F(u)\,du = \frac{5}{9}F\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}F(0) + \frac{5}{9}F\left(\sqrt{\frac{3}{5}}\right)]$$

$$= \frac{5}{9}\left[\frac{1}{\left(-\sqrt{\frac{3}{5}}\right)^2 + 2\left(-\sqrt{\frac{3}{5}}\right) + 5}\right] + \frac{8}{9}\left(\frac{1}{5}\right)$$

$$+ \frac{5}{9}\left[\frac{1}{\left(\sqrt{\frac{3}{5}}\right)^2 + 2\left(\sqrt{\frac{3}{5}}\right) + 5}\right]$$

$$= \frac{5}{9}\left[\frac{1}{\left(\frac{3}{5} + 5 - 2\sqrt{\frac{3}{5}}\right)} + \frac{1}{\left(\frac{3}{5} + 5 + 2\sqrt{\frac{3}{5}}\right)}\right] + \frac{8}{45}$$

$$= \frac{5}{9}\left[\frac{\left(\frac{28}{5} + 2\sqrt{\frac{3}{5}}\right) + \left(\frac{28}{5} - 2\sqrt{\frac{3}{5}}\right)}{\left(\frac{28}{5} - 2\sqrt{\frac{3}{5}}\right)\left(\frac{28}{5} + 2\sqrt{\frac{3}{5}}\right)}\right] + \frac{8}{45}$$

$$= \frac{5}{9}\left[\frac{56/5}{\left(\frac{28}{5}\right)^2 - 4\left(\frac{3}{5}\right)}\right] + \frac{8}{45}$$

$$= \frac{56}{9}\left[\frac{25}{(784-60)}\right] + \frac{8}{45}$$

$$= \frac{56}{9} \times \frac{25}{724} + \frac{8}{45}$$

$$= \frac{1400}{6516} + \frac{8}{45}$$

$$= 0.2148 + 0.1778$$

$$\int_{-1}^{1} f(u)\,du = 0.3926$$

Hen $I = \int_0^1 \frac{1}{1+x^2}\,dx = 2 \times 0.3926$

$$= 0.7852$$

Verification

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}(x)\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

$$= 0.7854$$

by one point:

$$\int_{-1}^{1} f(u)\,du = 2f(0) = \frac{2}{5}$$

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{4}{5} = 0.8$$

by two point :

$$\int_{-1}^{1} F(u)\,du = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\left[\left(-\frac{1}{\sqrt{3}}\right)^2 + 2\left(-\frac{1}{\sqrt{3}}\right) + 5\right]} + \frac{1}{\left[\left(-\frac{1}{\sqrt{3}}\right)^2 + 2\left(\frac{1}{\sqrt{3}}\right) + 5\right]}$$

$$= \frac{1}{\left(\frac{1}{3} - \frac{2}{\sqrt{3}} + 5\right)} + \frac{1}{\left(\frac{1}{3} + 5 + \frac{2}{\sqrt{3}}\right)}$$

$$= \frac{1}{\left(\frac{16}{3} - \frac{2}{\sqrt{3}}\right)} + \left(\frac{1}{\frac{16}{3} + \frac{2}{\sqrt{3}}}\right)$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{8}{3} - \frac{1}{\sqrt{3}}\right)} + \frac{1}{\left(\frac{8}{3} + \frac{1}{\sqrt{3}}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{16/3}{\left(\frac{8}{3}\right)^2 - \left(\frac{1}{3}\right)}\right]$$

$$= \frac{8}{3}\left[\frac{9}{64 - 3}\right]$$

$$= \frac{24}{61}$$

$$= 0.3934$$

$$I = \int_{0}^{1} \frac{1}{(1+x^2)}\,dx = 2\int_{-1}^{1} f(u)\,du = 0.7869$$

Unit - III

Numerical sol of algebraic & transcedental eqn

$$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \quad , \quad a_0 \neq 0$$

$P_n(x) = 0$ is called algebraic eqn

& Any eqn which contain exponential, trigonometric, logarithmic fn is called trancedental eqn

eg $\quad 7x^3 + 2e^x + 5\log(x+2) + 10\sin x = 0$

To find the root of eqn's we use

(i) Bisection method

(ii) Regula-falsi method

(iii) method of iteration

(iv) Newton's method.

(I) Bisection method:

let $x \in [a, b]$

when $f(a)f(b) < 0$

Now divide the interval $[a, b]$ in two parts by taking mid point $\left[\frac{a+b}{2}\right]$ say x_1

if $f(a)f(x_1) < 0$ then New intervals $\left[a, \frac{a+b}{2}\right]$

if $f(x_1)f(b) < 0$ then New inte is $\left[\frac{a+b}{2}, b\right]$

Continue this process untill we get two roots are same or repeate roots.

Q: Find the real root of eqⁿ $x^2 - 25 = 0$

Solⁿ Given eqⁿ is $x^2 - 25 = 0$ —— (1)

Let $f(x) = x^2 - 25$

$\because f(2) = -21$ & $f(7) = 24$

& $f(2)\, f(7) < 0$

$\therefore x \in [2, 7]$

Step-1 [First Approximation]

$$x_1 = \frac{2+7}{2} = 4.5$$

$$f(x_1) = (4.5)^2 - 25 = -4.75$$

$\because f(x_1)\, f(7) < 0$

$\therefore x \in [4.5, 7]$

Step-2 [Second Approximation]

$$x_2 = \frac{4.5+7}{2} = 5.75$$

$$f(5.75) = 8.0625$$

~~$\because f(5.75)\, f(2) < 0$~~

$\because f(4.5)\, f(5.75) < 0$

$\therefore x \in [4.5, 5.75]$

Third Approximation: $x_3 = \dfrac{4.5 + 5.75}{2} = 5.125$

$f(x_3) = \cancel{5.125}\ 1.2656$

$\because f(4.5)\, f(5.125) < 0$

$\therefore x \in [4.5, 5.125]$

step	a	b	xi
1	2	7	4.5
2	4.5	7	5.75
3	4.5	5.75	5.125
4	4.5	5.125	4.8125
5	4.8125	5.125	4.9687
6	4.9687	5.125	5.0469
7	4.9687	5.0469	5.0078
8	4.9687	5.0078	4.9882
9	4.9882	5.0078	4.9980
10	4.9980	5.0078	5.0029
11	4.9980	5.0029	5.0004
12	4.9980	5.0004	4.9992
13	4.9992	5.0004	4.9998
14	4.9998	5.0004	5.0001
15	4.9998	5.0001	4.9999
16	4.9999	5.0001	5.0000
17	4.9999	5.0000	5.0000

~~Hen~~ $\because x_{16} \cong x_{17}$ & $f(x_{17}) \cong 0$

$\therefore$ Real root of given is 5.0000 (Approx).

Regula-falsi method or method of false position or secant method.

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$\Rightarrow \quad x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_{n+1} \in [x_{n-1}, x_n]$$

$$\& \quad f(x_n) f(x_{n-1}) < 0$$

Q: Using Regula falsi method find the real root of the equation $x^4 + 2x^3 - x - 1 = 0$ in the interval $[0, 1]$.

Sol: Given eq is $x^4 + 2x^3 - x - 1 = 0$ — (1)

let $f(x) = x^4 + 2x^3 - x - 1$ — (2)

give interval, $[0,1] = [a, b]$

$f(a) = f(0) = -1$

$f(b) = f(1) = 1$

$\Rightarrow f(a) \cdot f(b) < 0$

$\Rightarrow x \in [0, 1]$

Regula falsi method

$$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)} \quad \text{— (3)}$$

First Approximation; $x_1 = \frac{0 f(1) - 1 \cdot f(0)}{f(1) - f(0)}$

$$= \frac{0 \times 1 - 1 \times (-1)}{1 - (-1)}$$

$$= \frac{1}{2}$$

$$f(x_1) = -1.19$$

$\because f(x_1) f(b) < 0$

$\therefore x \in [0.5, 1]$

Second Approximation: $x_2 = \dfrac{0.5 f(1) - 1 \cdot f(0.5)}{f(1) - f(0.5)}$

$= \dfrac{0.5 \times 1 - 1 \times (-1.19)}{1 - (-1.19)}$

$= \dfrac{0.5 + 1.19}{2.19}$

$= \dfrac{1.69}{2.19}$

$= 0.771689$

$\cong 0.7717$

$f(x_2) = -0.4979$

$\because f(x_2) f(b) < 0$

$\therefore x \in [0.7717, 1]$

Third Approximation: $x_3 = \dfrac{0.7717 \times f(1) - 1 \cdot f(0.7717)}{f(1) - f(0.7717)}$

$= \dfrac{0.7717 \times 1 - 1 \times (-0.4979)}{1 - (-0.4979)}$

$= \dfrac{0.7717 + 0.4979}{1.4979}$

$= \dfrac{1.2696}{1.4979}$

$= 0.84758$

$\cong 0.8476$

$f(x_3) = -0.11358 \cong 0.1136$

$\because f(0.8476) f(1) < 0$

$\therefore x \in [0.8476, 1]$

Approximation	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	0	1	-1	1	0.5	-1.19
2	0.5	1	-1.19	1	0.7717	-0.4979
3	0.7717	1	-0.4979	1	0.8476	-0.1136
4	0.8476	1	-0.1136	1	0.8631	-0.0222
5	0.8631	1	-0.0222	1	0.8661	-0.0013
6	0.8661	1	-0.0013	1	0.8663	-0.0023
7	0.8663	1	-0.0023	1	0.8666	-0.0010
8	0.8666	1	-0.0010	1	0.8667	-0.0004
9	0.8667	1	-0.0004	1	0.8667	-0.0004

$\because$ ~~Eighth & Ninth~~

$\because$ 8th & 9th Approximation are same

$\therefore$ one real root of the eq^n

$x^4 + 2x^3 - x - 1 = 0$ in interval $[0, 1]$

is 0.8667 (Approximate)

Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0$$

Q: find the real root of the eq^n $x^4 + 2x^3 - x - 1 = 0$

Sol: Given eq is $x^4 + 2x^3 - x - 1 = 0$

let $f(x) = x^4 + 2x^3 - x - 1$ —— (1)

& $f'(x) = 4x^3 + 6x^2 - 1$ —— (2)

To find real root of $f(x) = 0$ use

Newton Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

let initial root is $x_0 = 1$

then $f(x_0) = 1$, $f'(x_0) = 9$

first Approximation: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{1}{9}$

$= 0.8889$

$f(x_1) = 0.1401$

$f'(x_1) = 6.5503$

Second Approximation $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.8889 - \frac{0.1401}{6.5503}$

$= 0.8675$

$f(x_2) = \cancel{0.0040}\ 0.0045$

$f'(x_2) = 6.1267$

Third Approximation $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= \cancel{0.8675} - \cancel{\frac{0.0045}{6.1267}}$ $= 0.8675 - \frac{0.0045}{6.1267}$

$= 0.8342$ $= 0.8667$

Steps (x_n)	$f(x_n)$	$f'(x_n)$
$x_0 = 1$	1	9
$x_1 = 0.8889$	0.1401	6.5503
$x_2 = 0.8675$	0.0045	6.1267
$x_3 = 0.8667$	-0.0003	6.1112

$x_4 = 0.86674909 \cong 0.8667$

$\because x_3$ & x_4 are Approximately same

$\therefore$ one real root of the given eq $x^4 + 2x^3 - x - 1 = 0$

is $x = 0.8667$ (Approx)

Q. find the one real root of the eq $x^3 - 2x - 5 = 0$

Sol Given eq is $x^3 - 2x - 5 = 0$

let $f(x) = x^3 - 2x - 5$ —— (1)

~~Now~~ we use Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Now $f'(x) = 3x^3 - 2$ —— (2)

let $x_0 = 2$, $f(x_0) = -1$, $f'(x_0) = 10$

first Approx: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 2 - \frac{(-1)}{10} = 2.1$

~~f(x1)~~ $f(x_1) = 0.061$, $f'(x_1) = 11.23$

Secn Appr $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 2.1 - \frac{0.061}{11.23} = 2.0946$

$f(x_2) = 0.0005$, $f'(x_2) = 11.1620$

Thrd Appr $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 2.0946 - \frac{0.0005}{11.1620}$

$= 2.094555205$

$\cong 2.0946$

$\because x_2$ & x_3 are Approximately same

$\therefore$ one real root of the given eq $x^3 - 2x - 5 = 0$ is

$x = 2.0946$ (Approx)

Iteration method

$f(x) = 0 \Rightarrow \quad x = \phi(x)$

when $|\phi'(x)| < 1$, $\forall x \in [a, b]$

whe. $f(a) f(b) < 0$

Q: find the real root of the eq $x^3 - 2x - 5 = 0$

Sol Given eq is $x^3 - 2x - 5 = 0$ —— (1)

$f(x) = x^3 - 2x - 5$

$\because f(2) = -1$ & $f(3) = 16$ & $f(2) f(3) < 0$

$\therefore x \in [2, 3]$

$(1) \Rightarrow \quad x = (2x+5)^{1/3} = \phi(x)$ —— (2)

$$\phi'(x) = \frac{1}{3(2x+5)^{2/3}}$$

$\because |\phi'(x)| < 1, \ \forall x \in [2, 3]$

$\therefore$ ~~real root of~~ the eq (1) lie in the inter [2, 3]

$\therefore$ we can use iteration metho on $\phi(x)$.

$\Rightarrow \quad x_{n+1} = \phi(x_n)$

let $x_0 = 2$

$x_1 = \phi(x_0) = 3$

$x_2 = \phi(x_1) = 2.2240$

$x_3 = \phi(x_2) = 2.1140$

$x_4 = \phi(x_3) = 2.0975$

$x_5 = \phi(x_4) = 2.0950$

$x_6 = \phi(x_5) = 2.0946$

$x_7 = \phi(x_6) = 2.094558854 \cong 2.0946$

$\because x_6$ & x_7 are Approx same

$\therefore$ one real root of the eq $x^3 - 2x - 5 = 0$

is $x = 2.0946$ (Appr)

~~S: Solve~~

Q: Find the Approximate value of the smallest root of the eq^n

$x - \sin x - 1 = 0$

Sol^n Given eq^n is $x - \sin x - 1 = 0$ —(1)

let $f(x) = x - \sin x - 1$ — (2)

To find the root of eq^n (1) we use Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{—— (3)}$$

then

$$f'(x) = 1 - \cos x \quad \text{— (3)}$$

let initial root of eq^n (1) is

$$x_0 = 1$$

$$f(x_0) = -0.8415$$

$$f'(x_0) = 0.4597$$

First Approximation: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 1 - \frac{(-0.8415)}{0.4597}$$

$$= 2.8305$$

$$f(x_1) = 1.5244$$

$$f'(x_1) = 1.9520$$

Second Approximation: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 2.8305 - \frac{1.5244}{1.9520}$$

$$= 2.0496$$

$$f(x_2) = 0.1629$$

$$f'(x_2) = 1.4607$$

Third Approximation: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 2.0496 - \frac{0.1621}{1.4607}$$

$$= 1.9386$$

$$f(x_3) = 0.0463$$

$$f'(x_3) = 1.3596$$

Forth Appro.: $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$= 1.9386 - \frac{0.0463}{1.3596}$$

$$= 1.9045$$

$$f(x_4) = -0.0403$$

$$f'(x_4) = 1.3275$$

Fifth Appro. $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$

$$= 1.9045 - \frac{(-0.0403)}{1.3275}$$

$$= 1.9349$$

Approximation (x_n)	$f(x_n)$	$f'(x_n)$
$x_0 = 1$	-0.8415	0.4597
$x_1 = 2.8305$	1.5244	1.9520
$x_2 = 2.0496$	0.1621	1.4607
$x_3 = 1.9386$	0.0463	1.3596
$x_4 = 1.9045$	-0.0403	1.3275
$x_5 = 1.9349$	0.00045	1.3561
$x_6 = 1.9345$	-0.00009	1.3552
$x_7 = 1.9346$	0.000004	1.3558
$x_8 = 1.9346$	0.000004	1.3558

∵ 7th & 8th x_7 & x_8 are App. same
∴ the root of
given eq is
$x = 1.9346$

By Bisection Method

$$f(x) = x - \sin x - 1$$

$\because f(1) = -0.8415$

$f(2) = 0.0907$

$\because f(1)\,f(2) < 0$

$\therefore x \in [1, 2] = [a, b]$

First Appr.: $x_1 = \frac{1+2}{2} = 1.5$

$f(1.5) = -0.4975$

$\because f(1.5)\,f(2) < 0$

$\therefore x \in [1.5, 2]$

Second Appr.: $x_2 = \frac{1.5+2}{2} = 1.75$

$f(1.75) = -0.2340$

$\because f(1.75)\,f(2) < 0$

$\therefore x \in [1.75, 2]$

$\because$ ~~x_{14} & approx~~

x_{14} & x_{15} are Apr same

$\therefore$ real root of the equ,

$x = 1.9346$ Apr.

Approx.	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	1	2	-0.8415	0.0907	1.5	-0.4975
2	1.5	2	-0.4975	0.0907	1.75	-0.2340
3	1.75	2	-0.2340	0.0907	1.875	-0.0791
4	1.875	2	-0.0791	0.0907	1.9375	0.0040
5	1.875	1.9375	-0.0791	0.0040	1.9063	-0.0379
6	1.9063	1.9375	-0.0379	0.0040	1.9219	-0.0171
7	1.9219	1.9375	-0.0171	0.0040	1.9297	-0.0066
8	1.9297	1.9375	-0.0066	0.0040	1.9336	-0.0013
9	1.9336	1.9375	-0.0013	0.0040	1.9356	0.0014
10	1.9336	1.9356	-0.0013	0.0014	1.9346	0.00005
11	1.9336	1.9346	-0.0013	0.00005	1.9341	-0.0006
12	1.9341	1.9346	-0.0006	0.00005	1.9344	-0.0002
13	1.9344	1.9346	-0.0002	0.00005	1.9345	-0.00008
14	1.9345	1.9346	-0.00008	0.00005	1.9346	0.00005

By Regula falsi: $f(x) = x - \sin x - 1$ —— (1)

$\because f(1) = -0.8415$, $f(2) = 0.0907$

& $f(1) f(2) < 0$

$\therefore x \in [1, 2] = [a, b]$

$$x_{n+1} = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

first Approx: $x_1 = \frac{1 \cdot f(2) - 2 \cdot f(1)}{f(2) - f(1)} = \frac{0.0907 - 2 \times (-0.8415)}{0.0907 - (-0.8415)}$

$= \frac{1.7737}{0.9322}$

$= 1.9027$

$f(1.9027) = -0.0427$

$\because f(1.9027) f(2) < 0$

$\therefore x \in (1.9027, 2]$

Second Approx: $x_2 = \frac{1.9027 f(2) - 2 \cdot f(1.9027)}{f(2) - f(1.9027)}$

$= \frac{1.9027 \times 0.0907 + 2 \times 0.0427}{0.0907 + 0.0427}$

$= 1.9338$

$f(1.9338) = -0.0010$

$\because f(1.9338) f(2) < 0$

$\therefore x \in (1.9338, 2]$

Approx	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	1	2	-0.8415	0.0907	1.9027	-0.0427
2	1.9027	2	-0.0427	0.0907	1.9338	-0.0010
3	1.9338	2	-0.0010	0.0907	1.9345	-0.00009
4.	1.9345	2	-0.00009	0.0907	1.9346	0.00005
5	1.9345	1.9346	-0.00009	0.00005	1.9346	0.00005

$\because x_4$ & x_5 are Approx same

$\therefore$ one real root of the given eq (1)

$x = 1.9346$ (Approx

प्रश्नावली (Exercise) VIII

1. निम्न समीकरणों के न्यूनतम मूल का सन्निकटन मान लेखाचित्र विधि से ज्ञात कीजिए :

   Find the approximate value of the smallest root of the following equation using graphical method :

   (i) $x \log_{10} x = 1$ (ii) $x - \sin x - 1 = 0$ [Bikaner 13]

   (iii) $(x-1)^2 = \log_e x$

2. (a) द्विभाजन विधि से $x = 2$ तथा $x = 4$ के मध्य समीकरण $x^3 - 9x + 1 = 0$ का मूल ज्ञात कीजिए।

   Find the root of the equation $x^3 - 9x + 1 = 0$ between $x = 2$ and $x = 4$ by the method of bisection.

[Ajmer 03, 06, 13; Bikaner 04, 05, 13, 18; Kota 04, 06]

(b) द्विभाजन विधि द्वारा समीकरण $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ का वास्तविक मूल तीन दशमलव स्थानों तक ज्ञात कीजिए जो कि 2 तथा 3 के मध्य स्थित है।

Find the real root of the equation $x^4 - x^3 - 2x^2 - 6x - 4 = 0$, correct to three places of decimals by Bisection method which lies between 2 and 3.

(c) समीकरण $xe^x = 1$ के धनात्मक मूल के द्विभाजन विधि का प्रयोग कर ज्ञात कीजिए।

Using Bisection method, find a positve root of the equation $xe^x = 1$.

[Ajmer 07, 09]

(d) बाइसेक्शन विधि से (By bisection method) समीकरण का वास्तविक मूल ज्ञात कीजिए (find root of :) **[Alwar 18; Kota 17]**

(i) $x^3 - x - 1 = 0$ (ii) $3x - \sqrt{1 + \sin x} = 0$

3. मिथ्या–स्थिति विधि द्वारा समीकरण $x^6 - x^4 - x^3 - 1 = 0$ का 1.4 तथा 1.5 के मध्य वाला वास्तविक मूल चार दशमलव स्थानों तक ज्ञात कीजिए।

Find the real root between 1.4 and 1.5 to four decimal places of the equation $x^6 - x^4 - x^3 - 1 = 0$ by false position method.

[Bikaner 11; Ajmer 05]

4. मिथ्या स्थिति विधि द्वारा निम्न समीकरणों के सही तीन दशमलव स्थानों तक वास्तविक मूल ज्ञात कीजिए :

Find the real root of the following equations correct to three decimal places by Regular-Falsi method (Secant method):

(a) $x^3 - x^2 - 2 = 0$ **[Raj. 12]** (b) $x^3 - 4x + 1 = 0$ **[Kota 04]**

(c) $x^3 - 9x + 1 = 0$ **[Bikaner 11; Jodhpur 08, 16]**

(d) $x^3 + x - 1 = 0$, मध्य between 0 and 1 **[Ajmer 17]**

(e) $x^4 - x - 10 = 0$ **[Raj. 18]**

5. मिथ्या स्थिति विधि द्वारा निम्न समीकरणों के सही तीन दशमलव स्थानों तक वास्तविक मूल ज्ञात कीजिये :

Find the real root of the following equation correct to three decimal places by using Regular-Falsi method :

(i) $xe^x - 3 = 0$ (ii) $x^2 - \log_e x - 12 = 0$ **[Udaipur 06]**

(iii) $x \log_{10} x = 1.2$ **[Jodhpur 11; Bikaner 05, 18]**

(iv) $x^x = 100$, four decimal places. **[Ajmer 07]** (v) $x^2 - 6e^{-x} = 0$ **[Raj. 18]**

6. पुनरावृत्ति विधि द्वारा निम्न समीकरणों के वास्तविक मूल ज्ञात कीजिये :

Find the real root of the following equations by iteration method :

(i) $e^x - 3x = 0$ **[Jodhpur 13]** (ii) $\tan x = x$ **[Raj. 13]**

7. पुनरावृत्ति विधि द्वारा समीकरण $2x = \cos x + 3$ का मूल सही तीन दशमलव स्थानों तक ज्ञात कीजिये। **[Jodhpur 12]**

Find the root of the equation $2x = \cos x + 3$ correct to three decimal places by the method of iteration. [Raj. 14; Jodhpur 12]

8. $x = 0.12$ से प्रारम्भ करके पुनरावृत्ति विधि द्वारा समीकरण $x = 0.21 \sin(0.5 + x)$ को हल कीजिये।

Starting with $x = 0.12$, solve $x = 0.21 \sin(0.5 + x)$ by using iteration method.

9. न्यूटन–रेफसन विधि द्वारा निम्न समीकरणों का हल सही तीन दशमलव स्थानों तक ज्ञात कीजिये :

Use Newton-Raphson method to solve the following eqations correct to three decimal places :

(i) $2x - 5 = 3 \sin x$ (ii) $2(x-3) = \log_{10} x$, taking $x_0 = 3.2$ [Ajmer 07]

(iii) $x^3 - x - 2 = 0$. [Ajmer 14] (iv) $x^3 - \cos x = 0$ [Raj. 18]

10. न्यूटन–रेफसन विधि द्वारा निम्न समीकरणों का हल ज्ञात कीजिये :

By Newton-Raphson method, find the solution of the following equations :

(i) $x^3 - 3x + 4 = 0$ [Ajmer 17] (ii) $x^3 - x^2 - x - 1 = 0$

(iii) $x^3 - 25 = 0$ [Bikaner 12; Jodhpur 06]

11. न्यूटन–रेफसन विधि से $2x - 3\sin x - 5 = 0$ का वास्तविक मूल छः दशमलव स्थानों तक ज्ञात कीजिये।

Find the real root of $2x - 3 \sin x - 5 = 0$ upto six decimal places by Newton-Raphson method.

12. न्यूटन–रेफसन विधि द्वारा समीकरण $e^{-x} - \sin x = 0$ का न्यूनतम मूल चार दशमलव स्थानों तक ज्ञात कीजिये।

Find to four places of decimal the smallest root of the eqution $e^{-x} - \sin x = 0$ using Newton-Raphson method. [Raj. 13]

13. न्यूटन–रेफसन विधि द्वारा $\log x = \cos x$ को दशमलव के पांच स्थानों तक हल कीजिये।

Solve the equation $\log x = \cos x$ to five places of decimals by Newton-Raphson method. [Ajmer 2000; Kota 04]

14. मिथ्या–स्थिति विधि तथा न्यूटन–रेफसन विधि द्वारा समीकरण $3x - \cos x - 1 = 0$ को हल कीजिये।

प्रश्नावली (Exercise) VIII

1.	(i) 2.5	(ii) 1.9	(iii) 1, 1.8	
2.	(a) 2.9375	3. 1.4036	4. (a) 1.66	(b) 0.254
5.	(i) 1.051	(ii) 3.646	(iii) 2.7407	
6.	(i) 0.6190	(ii) 4.49s34	7. 1.5248, 0.12243	
9.	(i) 2.883238	(ii) 3.256	(iii) 1.521	
10.	(i) –2.1962	(ii) 1.8393	(iii) 2.924	
11.	2.883238		12. 0.5885	
13.	1.30295		14. 0.60710	
15.	3.4641			
16.	(i) 5.385165	(ii) 2.15466	(iii) 0.75	17. –1.2
19.	–1.225386		21. 2.74065	
22.	4.24			

Q: Find the real root of the equation $x^2 - \log_e x - 12 = 0$

Sol: Given eqⁿ is $x^2 - \log_e x - 12 = 0$ — (1)

let $f(x) = x^2 - \log_e x - 12$ — (2)

$\because f(3) = -4.0986,\ f(4) = 2.6137$

& $f(3)\,f(4) < 0$

$\therefore x \in [3, 4] = [a, b]$

(i) By Regula falsi' $x_n = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

first approximation: $x_1 = \dfrac{3f(4) - 4f(3)}{f(4) - f(3)} = \dfrac{24.2355}{6.7123} = 3.6106$

$f(x_1) = -0.2474$

$\because f(x_1)\,f(b) < 0$

$\therefore x \in [x_1, b]$

Second Approximation: $x_2 = \dfrac{3.6106\,f(4) - 4f(3.6106)}{f(4) - f(3.6106)} = \dfrac{10.4266}{2.8611} = 3.6443$

$f(3.6443) = -0.0122$

$\because f(x_2)\,f(b) < 0$

$\therefore x \in [x_2, b]$

Approximation	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	3	4	-4.0986	2.6137	3.6106	-0.2474
2	3.6106	4	-0.2474	2.6137	3.6443	-0.0122
3	3.6443	4	-0.0122	2.6137	3.6460	-0.0003
4	3.6460	4	-0.0003	2.6137	3.6460	-0.0003

$\because x_3$ & x_4 are Approx same

$\therefore$ one real root of the given eqⁿ is

$x = 3.6460$ (App.)

(ii) by Newton's Raphson method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(x) = x^2 - \log_e x - 12$

$f'(x) = 2x - \frac{1}{x}$

Use $x_0 = 3.5$

$f(x_0) = -1.0028$

$f'(x_0) = 6.7143$

First approximation: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.5 - \frac{(-1.0028)}{(6.7143)} = 3.6494$

$f(x_1) = 0.0236$

$f'(x_1) = 7.0248$

Second Appro: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.6494 - \frac{0.0236}{7.0248} = 3.6460$

$f(x_2) = -0.0003$

$f'(x_2) = 7.0177$

Third Appro: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.6460 - \frac{(-0.0003)}{7.0177} = 3.6460$

$\because$ x_2 & x_3 are Appro same

$\therefore$ one real root of the given eqn is $x = 3.6460$ (Appr

Q: Find $\sqrt{12}$.

Sol: let $\cancel{f(x)} = \sqrt{12} = x$

$\Rightarrow x^2 - 12 = 0$

~~let~~ $f(x) = x^2 - 12$ —(1)

To find $\sqrt{12}$ use Newton Raphson method : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f'(x) = 2x$

let $x_0 = 3.7$

$f(x_0) = 1.69$

$f'(x_0) = 7.4$

First Appro: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.7 - \frac{1.69}{7.4} = 3.4716$

$f(x_1) = 0.0520$

$f'(x_1) = 6.9432$

Second App: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.4716 - \frac{0.0520}{6.9432} = 3.4641$

$f(x_2) = -0.00001$

$f'(x_2) = 6.9282$

Third App: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4641 - \frac{(-0.00001)}{6.9282} = 3.4641$

$\because$ x_2 & x_3 are Appro same

$\therefore \sqrt{12} = 3.4641$ (App)

द्विघातीय है।

§ 8.17 एक दी हुई संख्या का p वां मूल ज्ञात करने के लिए न्यूटन का पुनरावृत्ति सूत्र (Finding p^th root of a given number by Newton's iterative formula)

किसी दी हुई राशि 'a' का p वां मूल समीकरण $x^p - a = 0$ का मूल लिया जा सकता है जिसे कि न्यूटन–रेफसन विधि द्वारा हल किया जा सकता है। **[Kota 16]**

मानलो $f(x) = x^p - a,$ तब $f'(x) = px^{p-1}$

अब न्यूटन पुनरावृत्ति सूत्र द्वारा

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \qquad x_{n+1} = x_n - \frac{(x_n)^p - a}{p(x_n)^{p-1}}$$

या
$$x_{n+1} = \frac{(p-1)(x_n)^p + a}{p(x_n)^{p-1}} \qquad \text{...(8.74)}$$

यह किसी दी हुई संख्या 'a' का p वां मूल निकालने के लिए अभीष्ट पुनरावृत्ति सूत्र है।

विशेष स्थितियां *(Particular Cases) :*

(1) दी हुई संख्या का प्रतिलोम *(Inverse of a given number) :*

[Ajmer 07; Bikaner 08]

सूत्र (8.74) में $p = -1$ रखने पर, संख्या a के प्रतिलोम के लिये पुनरावृत्ति सूत्र प्राप्त किया जा सकता है, जो कि निम्न रुप का होगा :

$$x_{n+1} = x_n(2 - ax_n) \qquad \text{...(8.75)}$$

(2) दी हुई संख्या का वर्गमूल *(Square of root of a given number):*

सूत्र (8.74) में $p = 2$ रखने पर, संख्या के वर्गमूल के लिये पुनरावृत्ति सूत्र प्राप्त किया जा सकता है, जो कि निम्न रुप का होगा :

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right) \qquad ...(8.76)$$

(3) दी हुई संख्या का प्रतिलोम वर्गमूल *(Inverse square root of a given number) :*

सूत्र (8.74) में $p=-2$ रखने पर, संख्या a के प्रतिलोम वर्गमूल के लिये पुनरावृत्ति सूत्र प्राप्त किया जा सकता है, जो कि निम्न रुप का होगा :

$$x_{n+1}=\frac{1}{2}x_n(3-a\,x_n^2) \qquad ...(8.77)$$

(4) दी हुई संख्या के p वें मूल का प्रतिलोम *(Reciprocal of the p^{th} root of a given number) :*

सूत्र (8.74) में p के स्थान पर $(-p)$ रखने पर, संख्या a के p वें मूल के प्रतिलोम के लिये पुनरावृत्ति सूत्र प्राप्त किया जा सकता है, जो कि निम्न रुप का होगा :

$$x_{n+1}=\frac{x_n}{p}[p+1-a(x_n)^p] \qquad ...(8.78)$$

टिप्पणी : सूत्र (8.78) में $p=1$ तथा 2 रखकर (8.75) तथा (8.77) प्राप्त किये जा सकते हैं।

व्याख्यात्मक उदाहरण (Illustrative Examples)

उदाहरण 17 : न्यूटन सूत्र द्वारा निम्न के मान चार दशमलव स्थानों तक ज्ञात कीजिये :

Evaluate the following by using Newton's formula upto four places of decimals :

(a) $\sqrt{8}$ (b) $(12)^{1/3}$ अथवा (or) $(20)^{1/3}$

[Udaipur 17; Ajmer 2000, 03, 17]

हल : (a) चूंकि $4<8<9 \Rightarrow \sqrt{4}<\sqrt{8}<\sqrt{9}$

$\Rightarrow 2<\sqrt{8}<3$

अतः $\sqrt{8}$ का मान 2 तथा 3 के मध्य स्थित होगा। यदि $\sqrt{8}$ का प्रारम्भिक सन्निकटन मान $x_0=2.5$ लें, तो न्यूटन के वर्गमूल सूत्र से

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right), \text{ जहां } a=8$$

उपर्युक्त में $n=0$ रखने पर इसका प्रथम सन्निकटन x_1 होगा :

$$x_1=\frac{1}{2}\left(x_0+\frac{8}{x_0}\right)=\frac{1}{2}\left(2.5+\frac{8}{2.5}\right)=2.85$$

अब, यदि $\sqrt{8}$ के उत्तरोत्तर संन्निकटन $x_2, x_3, \ldots$ हो, तो

$$x_2=\frac{1}{2}\left(x_1+\frac{8}{x_1}\right)=\frac{1}{2}\left(2.85+\frac{8}{2.85}\right)=2.8285$$

$$x_3=\frac{1}{2}\left(x_2+\frac{8}{x_2}\right)=\frac{1}{2}\left(2.8285+\frac{8}{2.8285}\right)$$

$$=2.8284.$$

स्पष्ट है कि $x_2 \approx x_3$, अतः $\sqrt{8}=2.8284$ **उत्तर**

(b) चूंकि $8<12<27 \Rightarrow (8)^{1/3}<(12)^{1/3}<(27)^{1/3}$

$$\Rightarrow \quad 2<(12)^{1/3}<3.$$

अतः $(12)^{1/3}$ का मान 2 तथा 3 के मध्य स्थित होगा। यदि $(12)^{1/3}$ का प्रारम्भिक सन्निकटन मान $x_0=2.5$ लें, तो न्यूटन के घनमूल सूत्र से $(p=3)$

$$x_{n+1}=\frac{2x_n^3+a}{3x_n^2}=\frac{1}{3}\left(2x_n+\frac{a}{x_n^2}\right), \text{ जहां } a=12$$

उपर्युक्त में $n=0$ रखने पर इसका प्रथम सन्निकटन x_1 होगा

$$x_1=\frac{1}{3}\left[2(2.5)+\frac{12}{(2.5)^2}\right]=2.3066$$

अब, यदि $(12)^{1/3}$ के उतरोतर सन्निक न $x_2, x_3, \ldots.$ हों, तो

$$x_2=\frac{1}{3}\left[2(2.3066)+\frac{12}{(2.3066)^2}\right]=2.2895$$

$$x_3=\frac{1}{3}\left[2(2.2895)+\frac{12}{(2.2895)^2}\right]=2.2895$$

स्पष्ट है कि $x_2 = x_3$, अतः $(12)^{1/3} = 2.2895$. **उत्तर**

उदाहरण 18 : न्यूटन सूत्र के प्रयोग से सिद्ध कीजिये कि N का वर्ग मूल निम्न पुनरावृत्ति सूत्र से प्राप्त किया जा सकता है :

Using Newton's formula, prove that square root of N can be obtained by the following recurrence formula :

$$x_{n+1} = x_n\left(1 - \frac{x_n^2 - N}{2N}\right)$$ **[Raj. 02, 06; Kota 07]**

अतः 26 का वर्ग मूल ज्ञात कीजिये ।

Hence find the square root of 26 .

हलः (I) मान लो $f(x) = 1 - \frac{N}{x^2}$, तब $f(x) = 0 \Rightarrow x^2 = N$

पुनः $$f'(x) = \frac{2N}{x^3}$$

न्यूटन सूत्र का प्रयोग करने पर

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{\left(1 - \frac{N}{x_n^2}\right)}{\left(\frac{2N}{x_n^3}\right)} = x_n - \left(\frac{x_n^2 - N}{x_n^2}\right)\left(\frac{x_n^3}{2N}\right)$$

$$\therefore \quad x_{n+1} = x_n\left(1 - \frac{x_n^2 - N}{2N}\right)$$

यह अभीष्ट सूत्र है।

(II) उपरोक्त में $N = 26, n = 0, x_0 = 5.0$ रखने पर

$$x_1 = 5\left(1 - \frac{25 - 26}{52}\right) = 5.096$$

इसी प्रकार

$$x_2 = 5.096\left[1 - \frac{(5.096)^2 - 26}{52}\right]$$

$$= 5.096[1 + 0.000592] = 5.099$$

अतः $\sqrt{(26)} \cong 5.099.$

उत्तर

उदाहरण 19 : प्रदर्शित कीजिये कि $N = AB$ का वर्गमूल निम्न से प्राप्त होता है :

Show tha the square root of $N = AB$ is given by :

$$\sqrt{N} = \frac{S}{4} + \frac{N}{S}$$

जहां *(where)* $S = A + B$.

हल : न्यूटन के वर्गमूल के सुत्रानुसार

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \qquad ...(i)$$

जहां $x_n, \sqrt{(AB)}$ का n वा सन्निकटन है।

(i) में n के स्थान पर $n + 1$ रखने पर

$$x_{n+2} = \frac{1}{2}\left(x_{n+1} + \frac{N}{x_{n+1}}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(x_n + \frac{N}{x_n}\right) + \frac{N}{\frac{1}{2}\{x_n + (N/x_n)\}}\right] \qquad \text{[(i) से]}$$

अब यदि प्रथम सन्निकटन $x_n \cong \sqrt{(AB)} \cong A$ लें, तब

$$x_n + \frac{N}{x_n} = A + \frac{AB}{A} = A + B = S \quad \text{(दिया है)},$$

तथा द्वितीय सन्निकटन

$$x_{n+1} = \frac{S}{2}$$