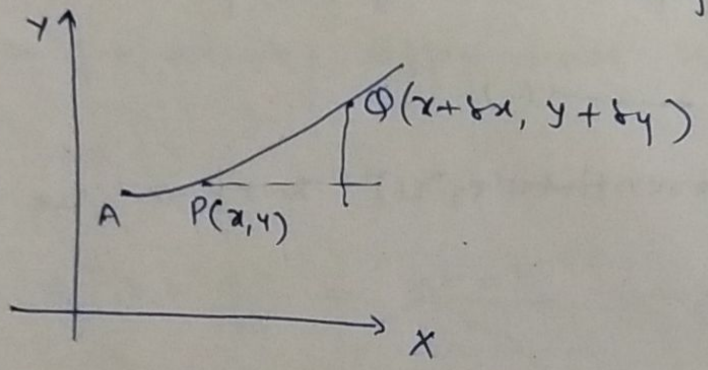
Velocity and ~~Accelat~~ Acceleration parallel to co-ordinate axes:

Let a particle move in XOY plane along the curve APQ where A is fixed point at the time t particle is at P(x, y) after some time (t + δt) particle is at Q(x + δx, y + δy) therefor displacement of particle along curve A $\vec{PQ}$



velocity along x axis:

$$V_x = \lim_{\delta t \to 0} \frac{\text{displacement along x axis in } \delta t \text{ time}}{\delta t}.$$

$$= \lim_{\delta t \to 0} \frac{\delta x}{\delta t}$$

$$\boxed{V_x = \frac{dx}{dt} = \dot{x}}$$

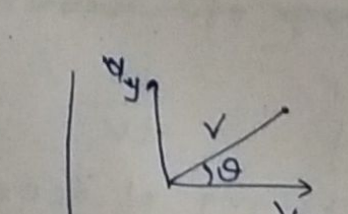
velocity along y axis:

$$V_y = \lim_{\delta t \to 0} \frac{\text{displacement along y axis in } \delta t \text{ time}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta y}{\delta t}$$

$$\boxed{V_y = \frac{dy}{dt} = \dot{y}}$$

Resultant velocity



$$V = \sqrt{V_x^2 + V_y^2}$$

direction of velocity is given by

$$\tan\theta = \frac{dy}{dx}$$

Q: The coordinate of a moving particle at time t are given by $x = a\cos^3 t$, $y = a\sin^3 t$, find its path, velocity & acceleration at time t.

Solution: The coordinate of a moving particle at time t are

$$x = a\cos^3 t \quad \text{---(1)}, \quad y = a\sin^3 t \quad \text{---(2)}$$

for path eliminate t, ~~for~~

$$\because \cos^2 t + \sin^2 t = 1$$

$$\therefore \left[\left(\frac{x}{a}\right)^{1/3}\right]^2 + \left[\left(\frac{y}{a}\right)^{1/3}\right]^2 = 1 \qquad \text{by (1) \& (2)}$$

$$\Rightarrow \boxed{x^{2/3} + y^{2/3} = a^{2/3}}$$

which is astroid.

for velocity differentiate eq^n (1) & (2) w.r.t t, we get

$$\frac{dx}{dt} = -3a\cos^2 t \sin t \quad \text{---(3)}$$

$$\frac{dy}{dt} = 3a\sin^2 t \cos t \quad \text{---(4)}$$

$$\because v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\therefore v = \sqrt{(9a^2\cos^4 t \sin^2 t + 9a^2\sin^4 t \cos^2 t)}$$

$$= 3a\sin t\cos t\sqrt{(\cos^2 t + \sin^2 t)}$$

$$\boxed{v = 3a\sin t\cos t}$$

for acceleration differentiate eq^n (3) & (4) w.r.t t, we get

$$\frac{d^2x}{dt^2} = -3a^2\cos^3 t + 6a\cos t\sin^2 t$$

$$\frac{d^2y}{dt^2} = -3a^2\sin^3 t + 6a\sin t\cos^2 t$$

$$\because f = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

$$\therefore f = \sqrt{(-3a\cos^3 t + 6a\cos t\sin^2 t)^2 + (-3a\sin^3 t + 6a\sin t\cos^2 t)^2}$$

$$= 3a\sqrt{\left[\cos^2 t(2\sin^2 t - \cos^2 t)^2 + \sin^2 t(2\cos^2 t - \sin^2 t)^2\right]}$$

$$= 3a\sqrt{\left[\cos^2 t(4\sin^4 t + \cos^4 t - 4\sin^2 t\cos^2 t) + \sin^2 t(4\cos^4 t + \sin^4 t - 4\cos^2 t\sin^2 t)\right]}$$

$$= 3a\sqrt{(4\cos^2 t\sin^4 t + \cos^6 t - 4\sin^2 t\cos^4 t + 4\sin^2 t\cos^4 t + \sin^6 t - 4\cos^2 t\sin^4 t)}$$

$$= 3a\sqrt{(\cos^6 t + \sin^6 t)}$$

$$= 3a\sqrt{[(\cos^2 t)^3 + (\sin^2 t)^3]}$$

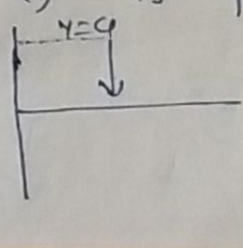
$$= 3a\sqrt{(\cos^2 t + \sin^2 t)(\cos^4 t - \cos^2 t\sin^2 t + \sin^4 t)}$$

$$= 3a\sqrt{[(\cos^2 t + \sin^2 t)^2 - 3\cos^2 t\sin^2 t]}$$

$$\boxed{f = 3a\sqrt{(1 - 3\cos^2 t\sin^2 t)}}$$

Q: A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is μy^{-2} and is initially projected with a velocity $\sqrt{(2\mu/a)}$ parallel to the axis of x at the point where $y = a$, prove that it will describe a cycloid.

Solution: The acceleration of moving particle parallel to the axis of y (alway towards the axis of x) is μy^{-2}



i.e. $\dfrac{d^2y}{dt^2} = -\mu y^{-2}$ —— (1)

and particle initialy projected with a velocity $\sqrt{\left(\frac{2\mu}{a}\right)}$ parallel to the axis of x at $y = a$

i.e at ~~x=r\theta~~, $y = a$ —(2)

$$\frac{dx}{dt} = \sqrt{\frac{2\mu}{a}} \quad —(3)$$

by (1) $\frac{d^2y}{dt^2}\left(2\frac{dy}{dt}\right) = -\frac{2\mu}{y^2}\left(\frac{dy}{dt}\right)$

$$\Rightarrow \int \frac{d}{dt}\left\{\left(\frac{dy}{dt}\right)^2\right\} dt = 2\mu\int \frac{d}{dt}\left(\frac{1}{y}\right) dt$$

$$\Rightarrow \left(\frac{dy}{dt}\right)^2 = 2\mu\left(\frac{1}{y}\right) + c_1 \quad —(4)$$

at $y = a$, $\frac{dy}{dt} = 0 \Rightarrow c_1 = -2\mu\left(\frac{1}{a}\right)$

$$(4) \Rightarrow \left(\frac{dy}{dt}\right)^2 = 2\mu\left(\frac{1}{y} - \frac{1}{a}\right)$$

$$\Rightarrow \frac{dy}{dt} = -\frac{2\mu}{a}\sqrt{\left(\frac{a-y}{y}\right)}$$ (towards the axis of x) —(5)

by (3) & (5)

$$\frac{dy}{dx} = -\sqrt{\frac{a-y}{y}}$$

$$\sqrt{\frac{y}{(a-y)}}\, dy = -dx \quad —(6)$$

let $y = a\cos^2\theta$ —(7)

$$dy = -2a\cos\theta\sin\theta\, d\theta$$

$$(6) \Rightarrow \sqrt{\frac{a\cos^2\theta}{(a - a\cos^2\theta)}}(-2a\cos\theta\sin\theta d\theta) = -dx$$

$$\Rightarrow dx = 2a\cos^2\theta\, d\theta$$

$$\Rightarrow dx = a(1+\cos 2\theta)\,d\theta$$

on intigration

$$x = a\left(\theta + \frac{\sin 2\theta}{2}\right) + c_2$$

initial when $y = a$, $\theta = 0$ by (7)

$x = 0$

we get $c_2 = 0$

$$\therefore \boxed{x = \frac{a}{2}(2\theta + \sin 2\theta)} \quad \text{— (8)}$$

$$\boxed{y = \frac{a}{2}(1+\cos 2\theta)} \quad \text{— (9) by (7)}$$

$\because$ eqn (8) + (9) are parametric eq of a cycloid

Hence particle will describe a cycloid.

Radial and Transverse Velocities and Accelerations:

Radial velocity: $\cancel{v = r\frac{d\theta}{dt}}$ $\quad u = \frac{dr}{dt}$

Transverse velocity: $v = r\frac{d\theta}{dt}$

Resultant velocity $= \sqrt{u^2+v^2} = \sqrt{\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2}$

Direction of velocity $(\alpha) \Rightarrow \tan\alpha = r\frac{d\theta}{dr} = \tan\phi$

Radial Acceleration: $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

Transverse Acceleration: $r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

Angular velocity: $\frac{d\theta}{dt}$

Angular Acceleration: $\frac{d^2\theta}{dt^2}$

Q: The radial and transversal velocities of a particle are λr and $\mu\theta$. Find its path and show that the radial and transverse component of accelerations are $\lambda^2 r - \frac{\mu^2\theta^2}{r}$ and $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$.

Solution: Given that

Radial velocity: $\cancel{\frac{dr}{dt} = \lambda r\theta}$ $\quad \frac{dr}{dt} = \lambda r$ — (1)

Transverse velocity: $r\frac{d\theta}{dt} = \mu\theta$ — (2)

for path eliminate t by (1) & (2), we get

$$r\frac{d\theta}{dr} = \frac{\mu\theta}{\lambda r}$$

$$\Rightarrow \frac{1}{r^2}\,dr = \frac{\lambda}{\mu}\cdot\frac{1}{\theta}\,d\theta$$

on integration

$$\boxed{-\frac{1}{r} = \frac{\lambda}{\mu}\log\theta + \text{conc } c_1}$$

Radial Acceleration : $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\left(\frac{d\theta}{dt}\right)^2$

$$= \frac{d}{dt}(\lambda r) - r\left(\frac{\mu\theta}{r}\right)^2 \quad \text{by (1) \& (2)}$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2 r\theta^2}{r^2}$$

$$= \cancel{\lambda^2 r - \mu^2\theta^2}$$

$$\boxed{\text{Radial Acc.} = \lambda^2 r - \frac{\mu^2\theta^2}{r}}$$

Transverse Acceleration : $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2\cdot\frac{\mu\theta}{r}\right)$ by (2)

$$= \frac{1}{r}\frac{d}{dt}(r\mu\theta)$$

$$= \frac{\mu}{r}\left[r\frac{d\theta}{dt} + \theta\frac{dr}{dt}\right]$$

$$= \frac{\mu}{r}\left[\mu\theta + \theta(\lambda r)\right]$$

$$\boxed{\text{Transver Acc.} = \mu\theta\left(\lambda + \frac{\mu}{r}\right)}$$

Q: A particle describes the curve $r = ae^{\theta}$ with constant angular velocity. show that its radial acceleration is zero and transverse acceleration varies as its distance from the pole.

Solution: Particle moving along the curve

$$r = ae^{\theta} \quad \text{—— (1)}$$

with constant anguler velocity

$$\text{i.e. } \frac{d\theta}{dt} = k \quad \text{—— (2)}$$

$$\text{Radial acceleration} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{d^2}{dt^2}(ae^{\theta}) - ae^{\theta}(k)^2$$

$$= a\frac{d}{dt}\left(e^{\theta}\frac{d\theta}{dt}\right) - ak^2e^{\theta}$$

$$= a\frac{d}{dt}(e^{\theta}.k) - ak^2e^{\theta}$$

$$= ak\frac{d}{dt}(e^{\theta}) - ak^2e^{\theta}$$

$$= ak\left(e^{\theta}\frac{d\theta}{dt}\right) - ak^2e^{\theta}$$

$$= ak^2e^{\theta} - ak^2e^{\theta}$$

$$\boxed{R.A = 0}$$

$$\text{Transeverse Acceleration} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$= \frac{1}{r}\frac{d}{dt}\left(a^2e^{2\theta}k\right)$$

$$= \frac{a^2k}{r}\frac{d}{dt}\left(e^{2\theta}\right)$$

$$= \frac{a^2k}{r}.2e^{2\theta}\frac{d\theta}{dt}$$

$$= \frac{2a^2k^2}{r}e^{2\theta} = \frac{2k^2}{r}(ae^{\theta})^2 = 2k^2r$$

$$\boxed{T.A \propto r}$$

Q: ~~A~~ particle P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversaley as its distances from O. show that the curve is an ~~eq~~ equiangular spiral.

Sol: Let the velocity of particle P (r, θ) is constant

i.e $V = a$ — ~~(1)~~

$$\Rightarrow \sqrt{(R.V) + (T.V)^2} = a$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = a^2 \text{ — (1)}$$

Angular velocity about ~~P(r,θ)~~ O varies inversely as its distance from O i.e $\frac{d\theta}{dt} \propto \frac{1}{r}$

$$\Rightarrow \frac{d\theta}{dt} = \frac{b}{r}$$

$$\Rightarrow r\frac{d\theta}{dt} = b \text{ — (2)}$$

which is transverse velocity

by (1) & (2)

$$\left(\frac{dr}{dt}\right)^2 = a^2 - b^2$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{a^2 - b^2} \text{ — (3)}$$

for curve eliminate dt by (2) & (3), we get

$$\frac{dr}{r\,d\theta} = \frac{\sqrt{a^2 - b^2}}{b} = c$$

$$\Rightarrow \frac{dr}{r} = c\,d\theta$$

on integration, we get

$$\log r = c\theta + \log K$$

$$\Rightarrow r = Ke^{c\theta}$$

which is an equiangular spiral.

Q: A particle moves along a circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero, prove that

$$\frac{d^2\theta}{dt^2} = -2\cot\theta\left(\frac{d\theta}{dt}\right)^2$$

sol. Particle moves along a circle $r = 2a\cos\theta$ —(1)

such that acceleration towards the origin ie. Radial acceleration is zero

ie. $$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 \quad \text{—(2)}$$

by (1) $$\frac{dr}{dt} = -2a\sin\theta\frac{d\theta}{dt}$$

$$\frac{d^2r}{dt^2} = -2a\sin\theta\frac{d^2\theta}{dt^2} - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

then (2) $\Rightarrow$
$$-2a\sin\theta\frac{d^2\theta}{dt^2} - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2 - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -2\cot\theta\left(\frac{d\theta}{dt}\right)^2$$

Q: A small bead slides with constant speed v on a smooth wire in the shape of the cardioid $r = a(1+\cos\theta)$, show that its radial acceleration is always constant and its angular velocity is $\frac{v}{2a}\sec\left(\frac{\theta}{2}\right)$.

sol: A small bead moves along the curve $r = a(1+\cos\theta)$ —(1) with constant speed v ie

$$(\text{Radial vel.})^2 + (\text{Trans. vel})^2 = v^2$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = v^2 \quad \text{—(2)}$$

$$\Rightarrow \left(-a\sin\theta\frac{d\theta}{dt}\right)^2 + \left[a(1+\cos\theta)\frac{d\theta}{dt}\right]^2 = v^2 \quad \text{by (1)}$$

$$\Rightarrow a^2\left[\sin^2\theta + (1+\cos\theta)^2\right]\left(\frac{d\theta}{dt}\right)^2 = v^2$$

$$\Rightarrow a^2\left[2+2\cos\theta\right]\left(\frac{d\theta}{dt}\right)^2 = v^2$$

~~$\Rightarrow$ angular velocity $\frac{d\theta}{dt} = v$~~

$$\Rightarrow 4a^2\cos^2\left(\frac{\theta}{2}\right)\left(\frac{d\theta}{dt}\right)^2 = v^2$$

then angular velocity $\frac{d\theta}{dt} = \frac{v}{2a}\sec\left(\frac{\theta}{2}\right)$ — (3)

Now Radial accelaration $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\left(\frac{d\theta}{dt}\right)^2$

$$= \frac{d}{dt}\left[-a\sin\theta\frac{d\theta}{dt}\right] - a(1+\cos\theta)\left(\frac{d\theta}{dt}\right)^2$$

$$= -a\frac{d}{dt}\left[\sin\theta \cdot \frac{v}{2a}\sec\left(\frac{\theta}{2}\right)\right] - a(1+\cos\theta)\left[\frac{v}{2a}\sec\left(\frac{\theta}{2}\right)\right]^2$$

$$= -\frac{v}{2}\frac{d}{dt}\left[\frac{2\sin\frac{\theta}{2}\cos\theta/2}{\cos\theta/2}\right] - 2a\cos^2(\theta/2)\left[\frac{v^2}{4a^2}\sec^2\left(\frac{\theta}{2}\right)\right]$$

$$= -v\frac{d}{dt}(\sin\theta/2) - \frac{v^2}{2a}$$

$$= -v\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\frac{d\theta}{dt} - \frac{v^2}{2a}$$

$$= -\frac{v}{2}\cos\left(\frac{\theta}{2}\right)\cdot\frac{v}{2a}\sec\left(\frac{\theta}{2}\right) - \frac{v^2}{2a}$$

$$= -\frac{v^2}{4a} - \frac{v^2}{2a} = \text{const} \quad \because v = \text{const}.$$

Hence Radial accelerat is alwaycont. &

angular velov is $\frac{v}{2a}\sec(\theta/2)$.

प्रश्नावली I (C)

1. यदि एक कण a त्रिज्या के वृत्त की परिधी पर गमन करता है, तो वृत्त के केन्द्र को ध्रुव मानते हुए कण का अरीय वेग लिखिए।

   If a particle is moving along the circle of radius a and centre of the circle be taken as pole, then write the radial velocity of the particle.

   [**Ans** : 0] **[Ajmer 04]**

2. अरिय एवं अनुप्रस्थ वेग की परिभाषा दीजिए।

   Define Radial and transversal velocities. **[Ajmer 03]**

3. किसी कण के अरीय एवं अनुप्रस्थ वेग λr तथा $\mu\theta$ है। इसका पथ ज्ञात करो तथा सिद्ध करो कि अरीय एवं अनुप्रस्थ त्वरण क्रमश: $\lambda^2 r - \frac{\mu^2\theta^2}{r}$ तथा $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$ है।

   The radial and transversal velocities of a particle are λr and $\mu\theta$. Find its path and show that the radial and transverse components of accelerations, are $\lambda^2 r - \frac{\mu^2\theta^2}{r}$ and $\mu\theta\left(\lambda + \frac{\mu}{r}\right)$.

   [Ajmer 2000, 06, 07, 08, 16; Raj. 04; 15; Kota 05; Udaipur 06; Bikaner 07; Jodhpur 07]

   [**Ans.** $1/r = -(\lambda/\mu)\log\theta + C$]

4. यदि एक कण इस प्रकार गमन करता है। कि उसका अरीय एवं अनुप्रस्थ वेग सदैव समान हो, तो इसके वक्र का समीकरण ज्ञात कीजिए। इसी के साथ यदि कण का कोणीय वेग अचर हो, तो अरीय और अनुप्रस्थ त्वरण के बारे में व्याख्या कीजिये।

   If a particle moves in such a way that its radial and transverse velocities are always equal, then find the equation of its path. If in addition the angular velocity of the particle is constant, then explain about radial and transverse acceleration. **[Kota 06]**

   [**Ans.** समान कोणिक सर्पिल (Equiangular Spiral)

5. कोई कण समानकोणिक सर्पिल $r = ae^{m\theta}$ में समान चाल से चलता हैं। इसके अरीय और अनुप्रस्थ वेग तथा त्वरण ज्ञात कीजिए।

   A particle describes equiangular spiral $r = ae^{m\theta}$ with constant speed. Find the radial and transverse components of its velocity and acceleration. **[Bikaner 15, 17; Ajmer 17]**

   $$\left[\textbf{Ans. } \frac{c}{\lambda}; \frac{c}{\lambda m}; -\frac{c}{\lambda^2 m^2}\left(\frac{1}{r}\right); \frac{c^2}{\lambda^2 m}\left(\frac{1}{r}\right), \text{ जहां } \lambda = \sqrt{\left(1 + \frac{1}{m^2}\right)}\right]$$

6. एक कण एकसमान कोणीय वेग से वक्र $r = ae^{\theta}$ में चलता है। सिद्ध करो कि इसका अरीय त्वरण शून्य होगा तथा अनुप्रस्थ त्वरण इसकी ध्रुव से दूरी के समानुपाती होगा।

A particle describes the curve $r = ae^{\theta}$ with constant angular velocity. Show that its radial acceleration is zero and transverse acceleration varies as its distance from the pole.

[Bikaner 04, 08; Kota 04; Jodhpur 06, 13; Udaipur 12; Ajmer 04, 05, 14]

7. एक कण समानकोणिक सर्पिल $r = ae^{\theta}$ में इस प्रकार गतिमान है कि इसका अरीय त्वरण शून्य हो। सिद्ध करो कि इसका कोणीय वेग अचर होगा तथा उसके वेग व त्वरण में से प्रत्येक का परिमाण r के समानुपाती होगा।

A particle describes equiangular spiral $r = ae^{\theta}$ in such a manner that its acceleration has no radial component. Prove that its angular velocity is constant and that the magnitude of its velocity and acceleration is each proportional to r. **[Ajmer 01]**

8. एक बिन्दु P किसी वक्र में अचर वेग से चलता है और इसका किसी दिये हुए स्थिर बिन्दु O के सापेक्ष कोणीय वेग इसकी O से दूरी का व्युत्क्रमानुपाती है। सिद्ध करो कि वक्र एक समानकोणिक सर्पिल है।

A particle P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as its distances from O. Show that the curve is an equiangular spiral.

[Jodhpur 06; Bikaner 12]

9. एक कण वृत्त $r = 2a \cos \theta$ के अनुदिश इस प्रकार गमन करता है कि मूल बिन्दु की ओर इसका त्वरण सदैव शून्य होता है। सिद्ध करो कि

A particle moves along a circle $r = 2a \cos \theta$ in such a way that its acceleration towards the origin is always zero. Prove that

$$\frac{d^2\theta}{dt^2} = -2 \cot \theta \left(\frac{d\theta}{dt}\right)^2$$ **[Raj. 02; Alwar 11]**

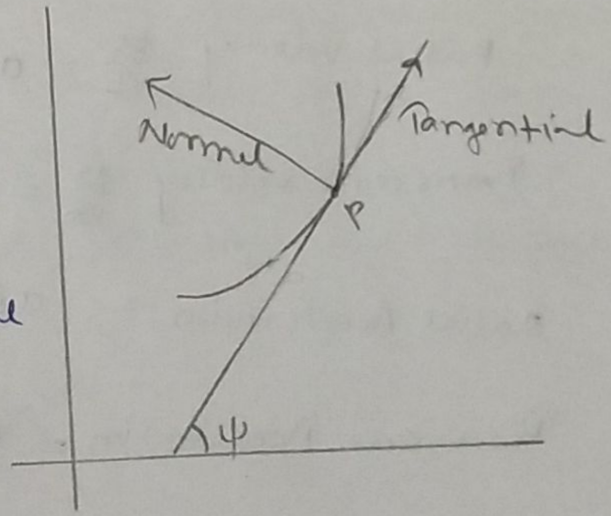
10. एक छोटा मणियां कार्डिऑइड $r = a\,(1+\cos \theta)$ के रूप वाले किसी चिकने तार पर अचर चाल v से फिसलता है। सिद्ध करो कि इसका अरीय त्वरण सदैव अचर रहता है तथा कोणीय वेग $\frac{v}{2a} \sec\left(\frac{1}{2}\theta\right)$ है।

A small bead slides with constant speed v on a smooth wire in the shape of the cardioid $r = a\ (1 + \cos \theta)$. Show that its radial acceleration is always constant and its angular velocity is $\frac{v}{2a}\sec\left(\frac{\theta}{2}\right)$

[Bikaner 06; Raj. 03, 06; Jodhpur 03, Ajmer 07, 09, 12; Kota 05, 13]

11. यदि एक बिन्दु किसी वृत्त में चलता हो तो सिद्ध करो कि उसकी परिधि पर किसी बिन्दु के सापेक्ष कोणीय वेग, केन्द्र के सापेक्ष कोणीय वेग का आधा होता है।

Tangential and Normal Velocity and Accelerations (Intrinsic form)



Tangential Velocity: Velocity along tangent at P $= \frac{ds}{dt} = v$

Normal Velocity: velocity along Normal at P $= 0$

~~Tangential Accel~~

Tangential Acceleration, Rate of change of velocity along tangent at P

$$= \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}, \quad \text{when } v = \frac{ds}{dt}$$

Normal Acceleration: Rate of change of velocity along Normal at P $= \frac{v^2}{\rho}$, when $v = \frac{ds}{dt}$ & ρ is radious of curvature

i.e $\rho = \frac{ds}{d\psi}$

Resultant Velocity $= \sqrt{(T.V)^2 + (N.V)^2}$

$$= \frac{ds}{dt}$$

Resultant Acceleration $= \sqrt{(T.A)^2 + (N.A)^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$

In circular path, $s = a\theta$, $\rho = a$

Radial velocity $\frac{dr}{dt} = 0 =$ Normal velocity

Transverse velocity $r\frac{d\theta}{dt} = a\dot{\theta} =$ Tangential velocity

Radial Acceleration $= a\dot{\theta}^2 =$ Normal Acceleration

Transverse Acceleration $= a\ddot{\theta} =$ Tangential Acceleration

Q: A particle moves in a curve so that its tangential and Normal accelerations are proportional to each other and the angular velocity of the tangent is constant. Find the curve.

Sol: $\because$ Tangential and Normal acceleration are proportional to each other & angular velocity of the tangent is constant.

$\therefore$ ~~$v\frac{ds}{ds} \propto \frac{v^2}{}$~~ $\therefore$ $v\frac{dv}{ds} \propto \frac{v^2}{\rho}$, $\frac{d\psi}{dt} = a$ (const)

$$\Rightarrow v\frac{dv}{ds} = k\frac{v^2}{\rho}$$

$$\Rightarrow \frac{1}{v}dv = k\,d\psi, \quad \because \rho = \frac{ds}{d\psi}$$

$$\Rightarrow \log v = k\psi + \log c$$

$$\Rightarrow v = ce^{k\psi}$$

$$\Rightarrow \frac{ds}{dt} = ce^{k\psi}$$

$$\Rightarrow \frac{ds}{d\psi}\cdot\frac{d\psi}{dt} = ce^{k\psi}$$

$$\Rightarrow ds = \frac{c}{a}e^{k\psi}d\psi \quad \Rightarrow s = \alpha e^{k\psi} + \beta$$

Q: If the tangential and normal acceleration of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any time t is given by $\rho = (at+b)^2$, where a & b are constants.

Sol: $\because$ Tangential and Normal Acceleration of a particle are constant

$$\therefore \quad \frac{d^2s}{dt^2} = c_1 \text{ (constant)}$$

$$\Rightarrow \quad v\frac{dv}{ds} = c_1 \quad \text{—— (1)}$$

and $$\frac{v^2}{\rho} = c_2 \quad \text{—— (2)}$$

$$\cancel{\Rightarrow v^2\frac{d\psi}{ds} = c_2 \to (2)} \qquad \because \cancel{\rho} = \frac{ds}{d\psi}$$

by (1) $\quad \cancel{v\,dv = c_1\,ds}$

$$\Rightarrow \cancel{\frac{v^2}{2} = c_1 s + c_3}$$

by (1) $$\frac{ds}{dt}\cdot\frac{dv}{ds} = c_1$$

$$\Rightarrow \frac{dv}{dt} = c_1$$

$$\Rightarrow dv = c_1\,dt$$

$$\Rightarrow v = c_1 t + c_3 \quad \text{—— (3)}$$

by (2) & (3)

$$\frac{(c_1 t + c_3)^2}{\rho} = c_2$$

$$\Rightarrow \rho = \frac{(c_1 t + c_3)^2}{c_2} = (at+b)^2$$

Q: A particle describes a cycloid $s = 4a\sin\psi$ with uniform speed v. Find its acceleration at any point. Also prove that the normal acceleration at any point varies inversely as the square root of the distance from the base of the cycloid.

Sol: ∵ Particle moves along $s = 4a\sin\psi$ —— (1)

with uniform speed v ie $\frac{ds}{dt} = v$ (const) —— (2)

∴ Tangential Acceleration $\frac{dv}{dt} = 0$ —— (3)

& Normal Acceleration $\frac{v^2}{\rho} = \frac{v^2}{\frac{ds}{d\psi}} = \frac{v^2}{4a\cos\psi}$ —— (4)

Resultant Acceleration $= \sqrt{(T.A)^2 + (N.A)^2}$

$$= \frac{v^2}{4a\cos\psi}$$

by (1) & (4)

$$N.A = \frac{v^2}{4a\sqrt{1-\sin^2\psi}} = \frac{v^2}{4a\sqrt{1-\frac{s^2}{16a^2}}}$$

$$= \frac{v^2}{\sqrt{(4a)^2 - s^2}}$$

$$\Rightarrow N.A \propto \frac{1}{\sqrt{(4a)^2 - s^2}}$$

Q. A point moves in a plane curve so that its tangential acceleration is constant and the magnitude of the tangential velocity and normal acceleration are in constant ratio. Show that the intrinsic equation of the path is of the form $s = A\psi^2 + B\psi + c$.

Sol. Particle move in a plane s.t

tangential acceleration is constant

i.e. $\frac{dv}{dt} = a$ — (1) or $v\frac{dv}{ds} = a$

The magnitude of the tangential velocity and Normal acceleration are in constant ratio

i.e. $$\frac{v}{\frac{v^2}{\rho}} = b$$

$$\Rightarrow \frac{\rho}{v} = b$$

$$\Rightarrow \frac{ds}{d\psi} \times \frac{dt}{ds} = b$$

$$\Rightarrow \frac{dt}{d\psi} = b$$

$$\Rightarrow t = b\psi + c_1 \quad — (2)$$

by (1) $$v = at + c_2$$

$$\Rightarrow \frac{ds}{dt} = at + c_2$$

$$\Rightarrow s = \frac{at^2}{2} + c_2 t + c_3 \quad — (3)$$

by (2) & (3)

~~s = At^2~~

$$\boxed{s = A\psi^2 + B\psi + c}$$

Q: A particle is moving in parabola with uniform angular velocity about the focus. Prove that its normal acceleration at any point is proportional to the radius of curvature of its path at the point.

Sol: Let the eq^n of parabola is $p^2 = ar$ —— (1)

$$\Rightarrow 2p\frac{dp}{dr} = a$$

$$\Rightarrow \rho = r\frac{dr}{dp} = \frac{2pr}{a} \quad \text{—— (2)}$$

particle move with uniform angular velocity

i.e $$\dot{\theta} = \frac{d\theta}{dt} = w \quad \text{—— (3)}$$

by the relation between angular & linear velocity

$$\dot{\theta} = \frac{vp}{r^2}$$

$$\Rightarrow v = \frac{wr^2}{p} \quad \text{—— (4)}$$

Normal acceleration $$\frac{v^2}{\rho} = \frac{\frac{w^2r^4}{p^2}}{\rho\,\cancel{\frac{2pr}{a}}} = \frac{w^2r^4}{ar}\cdot\frac{1}{\rho} \quad \text{by (1)}$$

$$= \frac{w^2r^3}{a}\cdot\frac{1}{\rho}$$

$$= \frac{w^2}{a^2}\cdot ar\cdot r^2\cdot\frac{1}{\rho}$$

$$= \frac{w^2}{a^2}\cdot p^2r^2\cdot\frac{1}{\rho} \quad \text{by (1)}$$

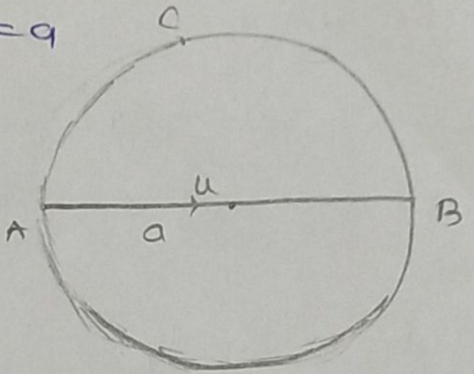
$$= \frac{w^2}{4}\cdot\frac{\rho^2}{\cancel{\rho}} \quad \text{by (2)}$$

$$= \frac{w^2}{4}\rho$$

$\boxed{N.A \propto \rho}$

Q: One point describes the diameter AB of a circle with constant velocity and another semi-circumference AB from rest with constant tangential acceleration. If they start together from A and arrive together at B. Show that the velocities at B are in the ratio $\pi : 1$

Sol: let radius of circle is a ie $OA = a$
Particle moves along diameter AB with constant velocity (u)



then time to arrive at B is $\boxed{t_1 = \frac{2a}{u}}$ —— (1)

if move along semi-circumference ACB with constant tangential acceleration

ie $v\frac{dv}{ds} = k$ —— (2)

$\Rightarrow \frac{v^2}{2} = ks + c_1$

at A, $s=0$, $v=0$, $c_1 = 0$

$\Rightarrow \boxed{v^2 = 2ks}$ —— (3)

at B $\boxed{v^2 = 2k\pi a}$ —— (3)

again by (2) $\frac{dv}{dt} = k$

$\Rightarrow v = kt + c_2$

at A, $t=0$, $v=0$, $c_2 = 0$

$\boxed{v = kt}$ —— (4)

at B $t_2 = \frac{v}{k} = \frac{v}{v^2/2\pi a} = \frac{2\pi a}{v}$ —— (5)

$\because$ they arrive at B together

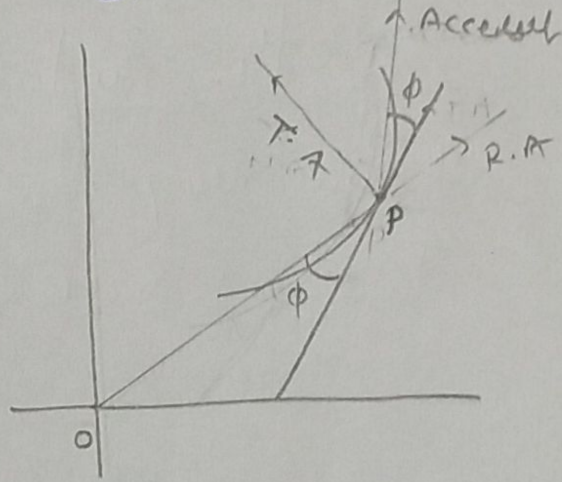
$\therefore t_1 = t_2$

$\frac{2a}{u} = \frac{2\pi a}{v} \Rightarrow \frac{v}{u} = \frac{\pi}{1} \Rightarrow \boxed{v : u = \pi : 1}$

Q: A particle P describe an equiangular spiral $r = a e^{\theta \cot \alpha}$ with constant angular velocity about the pole O. Show that its acceleration varies as OP and is in a direction making with the tangent at P the same constant angle that OP makes with the tangent.

Sol: Particle move along $r = a e^{\theta \cot \alpha}$ — (1)

with constant angular velocity i.e. $\frac{d\theta}{dt} = w$ — (2)



$\frac{dr}{dt} = a \cot \alpha\, e^{\theta \cos \alpha} \frac{d\theta}{dt}$

$= \cot\alpha (r w)$

$\frac{d^2 r}{dt^2} = w \cot\alpha \frac{dr}{dt}$

$= w^2 \cot^2\alpha \cdot r$

Radial Acceleration: $\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = w^2 \cot^2\alpha \cdot r - r w^2$

$= r w^2 (\cot^2\alpha - 1)$

Transverse Accelert: $\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}(w r^2)$

$= 2w \frac{dr}{dt}$

$= 2 r w^2 \cot\alpha$

Let accelr make an angle β with OP

then $\tan\beta = \frac{2 r w^2 \cot\alpha}{r w^2 (\cot^2\alpha - 1)} = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha$

$\Rightarrow \boxed{\beta = 2\alpha}$

angle between tangent & OP i.e. $\tan\phi = r\frac{d\theta}{dr} = \frac{rw}{rw\cot\alpha} = \tan\alpha$

$\phi = \alpha$ Hence $\boxed{\beta - \phi = \alpha}$

Q: A particle describes a curve (for which s and ψ vanish simultaneously) with uniform velocity v. If the acceleration at any point is $\frac{v^2 c}{s^2+c^2}$, find the intrinsic equation of the curve.

Sol. Particle move with uniform velocity v

i.e. $\frac{ds}{dt} = v$ —— (1)

$\Rightarrow \frac{d^2s}{dt^2} = 0$ —— (2)

The acceleration at any point is $\frac{v^2 c}{s^2+c^2}$

i.e. $\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2} = \frac{v^2 c}{s^2+c^2}$

$\Rightarrow \frac{v^2}{\rho} = \frac{v^2 c}{s^2+c^2}$

$\Rightarrow \frac{1}{\rho} = \frac{c}{s^2+c^2}$

$\Rightarrow \frac{d\psi}{ds} = \frac{c}{s^2+c^2}$ $\quad \because \rho = \frac{ds}{d\psi}$

$\Rightarrow d\psi = \frac{c}{s^2+c^2} ds$

on intigration

$\psi = \tan^{-1}\left(\frac{s}{c}\right) + K$

$\because$ when $s=0, \psi = 0 \Rightarrow K = 0$

$\therefore \psi = \tan^{-1}\left(\frac{s}{c}\right)$

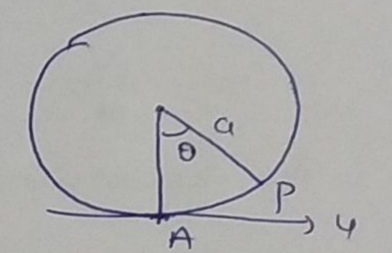
$\Rightarrow s = c\tan\psi$

which is catenary.

Q: A particle is describing a circle of radius a in such a way that the tangential acceleration is alway k times the normal acceleration. If its speed at a certain point is u, prove that it will return to the same point after a time $\frac{a}{ku}(1-e^{2\pi k})$

Sol: particle describe a circle of radious a.

~~from~~ particle move from the point A with ~~sp~~ velocity u and reach at the point P after time t



when $AP = s \Rightarrow s = a\theta, \ \rho = a$ —(1)

$\because$ Tangential Acceleration $= k \cdot$ (Normal Acceleration)

$$\Rightarrow v\frac{dv}{ds} = k\cdot\frac{v^2}{\rho}$$

$$\Rightarrow \cancel{v}\cdot\frac{dv}{d\theta}\cdot\frac{d\theta}{ds} = \frac{kv}{a} \qquad \text{by (1)}$$

$$\Rightarrow \frac{dv}{v}\cdot\frac{1}{a} = k\,d\theta$$

$$\Rightarrow \frac{1}{v}dv = k\,d\theta$$

on intigral

$$\log v = k\theta + c$$

$\because$ when $\theta = 0, v = u \Rightarrow c = \log u$

$\therefore \log v = k\theta + \log u$

$$\Rightarrow v = ue^{k\theta}$$

$$\Rightarrow \frac{ds}{dt} = ue^{k\theta}$$

$$\Rightarrow \frac{d\theta}{dt}\cdot\frac{ds}{d\theta} = ue^{k\theta}$$

$$\Rightarrow \frac{d\theta}{dt}\cdot a = ue^{k\theta}$$

$$dt = \frac{a}{u}e^{-k\theta}d\theta \quad —(2)$$

let particle take time T to return at A

when ~~then~~ $\theta = 0, t = 0$, ~~$\theta = 2\pi$~~,

$\theta = 2\pi, t = T$

on intigral of (2)

$$\int_0^T dt = \int_0^{2\pi}\frac{a}{u}e^{-k\theta}d\theta$$

$$T = \frac{a}{u}\left[\frac{e^{-k\theta}}{-k}\right]_0^{2\pi}$$

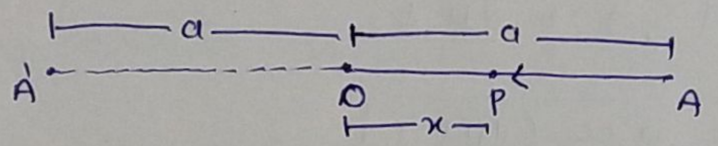
$$= -\frac{a}{ku}\left[e^{-2k\pi} - e^0\right]$$

$$T = \frac{a}{ku}\left[1 - e^{-2k\pi}\right]$$

Unit - III

Rectilinear Motion (S.H.M.)

Simple Harmonic Motion: When a particle moving in a straight line with an acceleration which is always towards a fixed point and varies as the distance from that point, then the motion of particle is called simple Harmonic motion and the fixed point is called centre of force.



~~eq^n of motion~~ $\cancel{m\frac{d^2x}{dt^2}} =$

$$\because \quad \frac{d^2x}{dt^2} \propto x$$

$\therefore$ eq of motion is $\quad m\frac{d^2x}{dt^2} = -m\mu x \quad$ — (1)

for v & x

$$v\frac{dv}{dx} = -\mu x$$

on intigration

$$\Rightarrow \frac{v^2}{2} = -\frac{\mu x^2}{2} + c_1$$

at A, $x = a$, $v = 0 \Rightarrow c_1 = \frac{\mu a^2}{2}$

$$\Rightarrow \frac{v^2}{2} = -\frac{\mu x^2}{2} + \frac{\mu a^2}{2}$$

$$\Rightarrow \boxed{v^2 = \mu(a^2 - x^2)} \quad \text{— (2)}$$

when particle move form A to O :

$$v = \frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2-x^2}$$

$$\Rightarrow \frac{-1}{\sqrt{a^2-x^2}}\,dx = \sqrt{\mu}\,dt$$

on intigration

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}t + c_2$$

at A, $x = a$, $t = 0$, $\Rightarrow c_2 = \cos^{-1}(1) = 0$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu}\,t$$

$$\Rightarrow \boxed{x = a\cos(\sqrt{\mu}t)}$$

when particle move from O to A :

$$v = \frac{dx}{dt} = \sqrt{\mu}\sqrt{a^2-x^2}$$

$$\Rightarrow \frac{1}{\sqrt{a^2-x^2}}\,dx = \sqrt{\mu}\,dt$$

$$\Rightarrow \sin^{-1}(x/a) = \sqrt{\mu}t + c_3$$

at O, $x = 0$, $t = 0 \Rightarrow c_3 = 0$

$$\Rightarrow \sin^{-1}(x/a) = \sqrt{\mu}\,t$$

$$\Rightarrow \boxed{x = a\sin(\sqrt{\mu}t)}$$

Remarks:

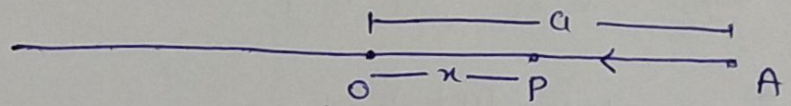
(i) $OA = A'O = a$ is called Amplitude of motion

(ii) ~~Re~~ Period of S.H.M : $T = 2\pi/\sqrt{\mu}$

Q: A particle is moving in SHM of amplitude a, at what distance from the centre will its velocity be half of the maximum?

Sol: If amplitude of SHM is a and at the time t particle is at the distance x from centre ~~the~~ velocity is v then

$$v^2 = \mu(a^2 - x^2) \quad \text{—— (1)}$$



velocity will be maximum if $x = 0$

ie $\quad v = \sqrt{\mu}\, a \quad$ —— (2)

Let at the distance x_1 velocity of particle be half of the maximum velocity is $v_2 = \frac{\sqrt{\mu}\, a}{2}$

then by (1)

$$\frac{\mu a^2}{4} = \mu(a^2 - x_1^2)$$

$$\Rightarrow \frac{a^2}{4} = a^2 - x_1^2$$

$$\Rightarrow x_1^2 = \frac{3a^2}{4}$$

$$\Rightarrow \boxed{x_1 = \frac{\sqrt{3}\, a}{4}}$$

Q: A particle is moving with SHM find the period ~~of~~ to move from the position of maximum displacement to one in which the displacement is half the amplitude.

Sol: Let the amplitude of SHM is a and at the time t distance x from centre, then

$$x = a\cos(\sqrt{\mu}t) \quad \text{---(1)}$$

if $x = a/2$ then by (1)

$$\frac{a}{2} = a\cos(\sqrt{\mu}t)$$

$$\Rightarrow \cos(\sqrt{\mu}t) = \frac{1}{2}$$

$$\Rightarrow \sqrt{\mu}t = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\Rightarrow t = \frac{\pi}{3\sqrt{\mu}} = \frac{2\pi}{6\sqrt{\mu}}$$

$\because$ Period of SHM is $T = 2\pi/\sqrt{\mu}$

$$\therefore \boxed{t = \frac{T}{6}}$$

Q: A body moving with S.H.M has an amplitude a and time period T. show that the velocity v at a distance x from the mean position is given by $v^2T^2 = 4\pi^2(a^2 - x^2)$

(i) If the velocity is trebled, when the distance from the mean position is $(2/3)a$, the period being unaltered, find the new amplitude.

(ii) If the velocity is doubled, where the distance from the mean position is $(a/2)$. The period being unaltered, find the new amplitude.

Sol: Given that $v^2T^2 = 4\pi^2(a^2 - x^2)$ ——— (1)

where T is time period & a is amplitude

(i) at $x = \frac{2}{3}a$, ~~$v_1 = 3v$~~ $v = v_1$

by (1) $$v_1^2 T^2 = 4\pi^2\left(a^2 - \frac{4a^2}{9}\right)$$

$$= \frac{20\pi^2 a^2}{9}$$

$$v_1^2 = \frac{20\pi^2 a^2}{9T^2}$$

if $v_1 \to 3v_1$ then amplitude is A

by (1) $$9v_1^2 T^2 = 4\pi^2\left(A^2 - \frac{4a^2}{9}\right)$$

$$\Rightarrow 20\pi^2 a^2 = 4\pi^2\left(A^2 - \frac{4a^2}{9}\right)$$

$$\Rightarrow 5a^2 + \frac{4a^2}{9} = A^2$$

$$\Rightarrow A^2 = \frac{49a^2}{9}$$

$$\Rightarrow A = \frac{7}{3}a$$

Hence New amplitude is $(7/3)a$.

(ii) at $x = \frac{a}{2}$, $V = V_2$

by (1)

$$V_2^2 T^2 = 4\pi^2 \left(a^2 - \frac{a^2}{4}\right)$$

$$\Rightarrow V_2^2 T^2 = 3\pi^2 a^2$$

if $V_2 \to 2V_2$ then New amplitude is B

by (1)

$$4V_2^2 T^2 = 4\pi^2 \left(B^2 - \frac{a^2}{4}\right)$$

$$\Rightarrow 3\pi^2 a^2 = \pi^2 \left(B^2 - \frac{a^2}{4}\right)$$

$$\Rightarrow B^2 = 3a^2 + \frac{a^2}{4}$$

$$\Rightarrow B^2 = \frac{13a^2}{4}$$

$$\Rightarrow B = \left(\frac{\sqrt{13}}{2}\right) a$$

Hence New amplitude is $\left(\frac{\sqrt{13}}{2}\right) a$.

7. एक कण स. आ. ग. में गतिमान है। एक सिरे से केन्द्र को जाते हुए यह पाया गया कि लगातार तीन सैकण्डों पर कण की केन्द्र से दूरी x_1, x_2, x_2 है। प्रदर्शित कीजिये कि एक पूर्ण आवर्त काल $\frac{2\pi}{\theta}$ है; जहां $\cos\theta = \frac{x_1 + x_3}{2x_2}$.

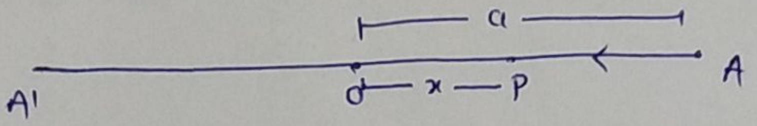
A particle is moving with S.H.M. from an extremity of path towards the centre is observed to be at distances x_1, x_2, x_3 from the centre at the end of three successive seconds. Show that the time of acomplete oscillation is $\frac{2\pi}{\theta}$, where $\cos\theta = \frac{x_1 + x_3}{2x_2}$.

[Ajmer 2000, 04, 07, 13, 16; Kota 06; Bikaner 04, 07, 11; Jodhpur 08; Raj. 05, 12, 14; Hons. 07; Udaipur 12]

8. एक कण का वेग v जो कि OX अक्ष के अनुदिश चल रहा है

Q2 If amplitude of SHM is a and at the time t particle is at the distance x from centre O then

$$x = a\cos(\sqrt{\mu}\,t) \quad \text{---}(1)$$



$\because$ Particle is at distances x_1, x_2, x_3 from the centre at the end of three successive seconds.

i.e. at $t = 1,\ x = x_1$

$t = 2,\ x = x_2$

$t = 3,\ x = x_3$

$\therefore$ by (1)

$$x_1 = a\cos(\sqrt{\mu}) \quad \text{---}(2)$$

$$x_2 = a\cos(2\sqrt{\mu}) \quad \text{---}(3)$$

$$x_3 = a\cos(3\sqrt{\mu}) \quad \text{---}(4)$$

by (2) & (4)

$$x_1 + x_3 = a\left[\cos(\sqrt{\mu}) + \cos(3\sqrt{\mu})\right]$$

$$= a\,2\cos(2\sqrt{\mu})\cos(\sqrt{\mu})$$

$$x_1 + x_3 = 2x_2\cos(\sqrt{\mu})$$

$$\Rightarrow \cos(\sqrt{\mu}) = \frac{x_1 + x_3}{2x_2}$$

$$\Rightarrow \sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$$

time of complete oscillation is $T = \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\theta}$

where $\cos\theta = \dfrac{x_1 + x_3}{2x_2}$

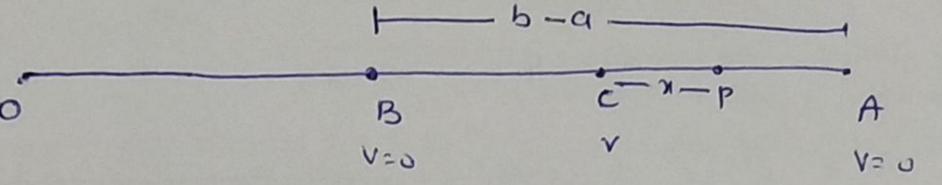
k 2 after starts of the first particle, provided $t < 2\pi/k$.

6. एक पिण्ड एक सरल रेखा OAB पर स. आ. ग. से गतिमान है। यह A तथा B पर विरामावस्था में है जिसकी O से दूरी क्रमशः a तथा b है और उसका वेग v है जब वह उनके मध्य बिन्दु पर है। प्रदर्शित कीजिए कि पूर्ण आवर्त काल $\dfrac{\pi(b-a)}{v}$ है।

A body moving in a straight line OAB with S.H.M. has zero velocity when at points A and B whose distances from O are a and b respectively and has a velocity v when half way between them. Show that the complete period is $\dfrac{\pi(b-a)}{v}$.

[Raj. 04, 16; Ajmer 02, 04, 07, 09, 12; Kota 05, 07, 14, Bikaner 13, 17; Jodhpur 13, 14]

Q. 6 A particle moving in straight line with SHM has zero velocity at A & B, when OA = a, OB = b



the amplitude of SHM is $\frac{b-a}{2}$ and centre of force is C where velocity is v.

let at the distance x from C velocity of particle is v_1

then
$$v_1^2 = \mu\left[\left(\frac{b-a}{2}\right)^2 - x^2\right]$$

at C, $x = 0$, $v_1 = v$

$$v^2 = \mu\left[\left(\frac{b-a}{2}\right)^2\right]$$

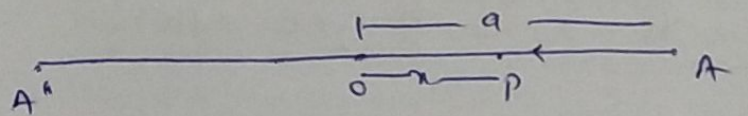
$$\sqrt{\mu} = \frac{v}{\left(\frac{b-a}{2}\right)} = \frac{2v}{b-a}$$

Then the complete period is $T = \frac{2\pi}{\sqrt{\mu}} = \frac{\pi(b-a)}{v}$

4. एक सरल रेखा में स.आ.ग. से दोलन करने वाले पिण्ड के इसके पथ के दो बिन्दुओं पर वेग u, v तथा त्वरण α तथा β हैं, तो प्रदर्शित करो कि इन दोनों बिन्दुओं के बीच की दूरी $\frac{v^2 - u^2}{\alpha + \beta}$, गति का आयाम $\frac{\sqrt{[(v^2 - u^2)(\alpha^2 v^2 - \beta^2 u^2)]}}{\beta^2 - \alpha^2}$ तथा आवर्तकाल $2\pi\sqrt{\frac{u^2 - v^2}{\beta^2 - \alpha^2}}$ होगा।

A point executes S.H.M. such that in two of its position the velocities are u, v and the corresponding accelerations are α, β; show that the distance between the positions is $\frac{v^2 - u^2}{\alpha + \beta}$, the amplitude of the motion is $\frac{\sqrt{[(v^2 - u^2)(\alpha^2 v^2 - \beta^2 u^2)]}}{\beta^2 - \alpha^2}$ and the time period is $2\pi\sqrt{\frac{u^2 - v^2}{\beta^2 - \alpha^2}}$. **[Kota 04, 08; Jodhpur 02, 05]**

Q. 4 Let particle moving is straight line with SHM, an amplitude is a and the velocity of particle is at the distance x from centre O is V_1 ie.



then $$\frac{d^2x}{dt^2} = -\mu x \quad \text{—— (1)}$$

$$V_1^2 = \mu(a^2 - x^2) \quad \text{—— (2)}$$

$\because$ at $x = x_1$, $V_1 = u$ & $\frac{d^2x}{dt^2} = \alpha$

and $x = x_2$, $V_1 = v$ & $\frac{d^2x}{dt^2} = \beta$

by (1) $\alpha = -\mu x_1$ — (3)

$\beta = -\mu x_2$ — (4)

by (2) $u^2 = \mu(a^2 - x_1^2)$ — (5)

$v^2 = \mu(a^2 - x_2^2)$ — (6)

by (5) & (6)

$$u^2 - v^2 = \mu(x_2^2 - x_1^2)$$

$$= \mu\left(\frac{\beta^2}{\mu^2} - \frac{\alpha^2}{\mu^2}\right) \quad \text{by (3) \& (4)}$$

$$= \frac{\beta^2 - \alpha^2}{\mu}$$

$$\Rightarrow \mu = \frac{\beta^2 - \alpha^2}{u^2 - v^2} \quad \text{— (7)}$$

the time period is $T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\frac{u^2 - v^2}{\beta^2 - \alpha^2}}$

again by (3) & (4)

$$\alpha - \beta = \mu(x_2 - x_1)$$

$$\Rightarrow x_2 - x_1 = \frac{\alpha - \beta}{\mu} = \frac{\alpha - \beta}{\frac{(\beta^2 - \alpha^2)}{(u^2 - v^2)}}$$

the distance between the position is $\frac{v^2 - u^2}{\alpha + \beta}$

by (5) $$a^2 = \frac{u^2}{\mu} + x_1^2 = \frac{u^2}{\mu} + \frac{\alpha^2}{\mu^2}$$

$$= \frac{u^2\mu + \alpha^2}{\mu^2}$$

$$= \frac{u^2\left(\frac{\beta^2-\alpha^2}{u^2-v^2}\right) + \alpha^2}{\left(\frac{\beta^2-\alpha^2}{u^2-v^2}\right)^2} \quad \text{by (7)}$$

$$= \frac{\left[u^2(\beta^2-\alpha^2) + \alpha^2(u^2-v^2)\right](u^2-v^2)}{(\beta^2-\alpha^2)^2}$$

$$a^2 = \frac{(u^2\beta^2 - \alpha^2 v^2)(u^2-v^2)}{(\beta^2-\alpha^2)^2}$$

$\Rightarrow$ amplitud $$a = \frac{\sqrt{(v^2-u^2)(\alpha^2 v^2 - \beta^2 u^2)}}{(\beta^2-\alpha^2)}$$

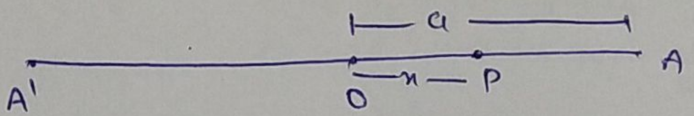
5. एक कण विरामावस्था से त्वरण k^2x के अधीन एक नियत बिन्दु की दिशा में चलना प्रारम्भ करता है तथा t समय पश्चात् दूसरा कण उसी बिन्दु से उसी त्वरण के अधीन उसी प्रकार चलना प्रारम्भ करता है। प्रदर्शित करो कि कण आपस में, पहले कण के गति करने के $\frac{\pi}{k} + \frac{t}{2}$ समय के पश्चात् टकरायेंगे, यदि $t < 2\pi/k$.

A particle starts from rest under an acceleration k^2x directed towards a fixed point and after time t another particle starts from the same positions under the same acceleration. Show that the particle will collide at time $\frac{\pi}{k} + \frac{t}{2}$ after starts of the first particle, provided $t < 2\pi/k$.

6. एक पिण्ड एक सरल रेखा OAB पर स

Q.5 Let A particle move ~~in~~ with SHM under an acceleration k^2x towards a fixed point O

ie.



$$\frac{d^2x}{dt^2} = k^2x \quad \text{——} \quad (1)$$

let $OA = a$ & $OP = x$

then $x = a\cos kt$ —— (2)

After time t another particle starts from the same position under the same acceleration when $t < 2\pi/k$

i.e second particle will starts their motion befor completing the period of first particle

let they collide at time t'

then by (1)

for first particle $x = a\cos kt'$ — (2)

& for second particle $x = a\cos k(t'-t)$ — (3)

$= a\cos[2\pi - k(t'-t)]$

by (2) & (3)

$$kt' = k(t'-t) \text{ or } kt' = 2\pi - k(t'-t)$$

$$\Rightarrow t = 0 \quad \text{or} \quad t' = \frac{\pi}{k} + t/2$$

$t = 0$ is not posible

$\therefore$ they collid at time $\frac{\pi}{k} + t/2$ after starts of the first particle.