Unit - 5
Electricfield

Coulomb's law

In 1785 Coulomb stated a law regarding the force acting b/w two charges:- The force of attraction or repulsion b/w two charges (point charges) is directly proportional to the product of the charges and inversely proportional to the square of distance b/w them. The direction of this force is along the line joining the two charge. This is ~~called~~ also called Coulomb's inverse square law.

q_1 .... r .... q_2

Let q_1 & q_2 are two point charges and separated by a distance r then force is

$$F \propto \frac{q_1 q_2}{r^2}$$

$$F = k\frac{q_1 q_2}{r^2} \quad \text{— ①}$$

where k is proportionality constant and depends on
(i) medium b/w the charges
(ii) system of unit chosen.

If the medium b/w charges is air or vacuum then value of k in MKSA unit system or SI unit is

$k = \frac{1}{4\pi\epsilon_0}$ where ϵ_0 is universal constant called permitivity of vacuum or free space and is given by

$$\epsilon_0 = 8.854\times10^{-12}\ C^2/Nm^2$$

$$\therefore k = \frac{1}{4\pi\epsilon_0} = \frac{1}{4\times3.14\times8.854\times10^{-12}}$$

$$k = 9\times10^{9}\ Nm^2/C^2$$

therefore coulomb's law

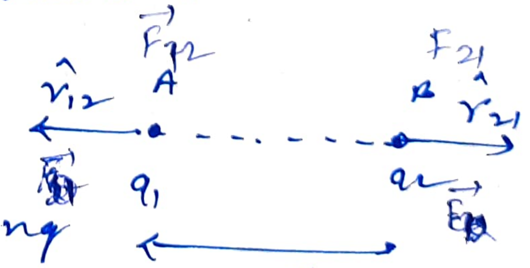
$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}$$

the magnitude of value of k ~~in~~ cgs unit system is on

the Coulomb's law in vector form

let two charges q_1 and q_2 are placed at points A and B in vacuum respectively and separated by distance r the charges are like, they repel each other.

let $\vec{F}_{12}$ is force acting on q_1 due to q_2 and $\vec{F}_{21}$ is force acting on q_2 due to q_1 and ~~xx~~ $\vec{r}_{12}$ is ~~position~~ unit ~~unit~~ ~~position~~ vector of q_1 w.r.to q_2 and $\vec{r}_{21}$ is ~~position~~ vector of q_2 w.r.to q_1. $\hat{r}_{12}$ and $\hat{r}_{12}$ is ~~position~~ unit vector along B to A and A to B respectively. the force $\vec{F}_{21}$ along A to B is



$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\hat{r}_{21}$$

but $\hat{r}_{21} = \frac{r_{21}}{r}$

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \cdot \frac{\vec{r}_{21}}{r}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^3} \vec{r}_{21} \quad —(2)$$

Similarly force on q_1 is

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^3} \hat{r}_{12} \quad —(3)$$

but $\vec{r}_{12} = -\vec{r}_{21}$ then

$$F_{12} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^3} \vec{r}_{21} \quad —(4)$$

then $F_{12} = -F_{21}$

this means that the Coulomb's force acted on q_2 by q_1 is equal ~~to~~ and opposite to the coulomb's force acted on q_1 by q_2 ~~to~~ according to Newton's third law.

<u>Unit of charge:-</u> Coulomb's

the S.I unit of charge is coulomb and denoted by 'C'

The force b/w two charges q_1 and q_2 and separated by r in air is

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

If $q_1 = q_2 = 1C$

$r = 1m$

then $F = \frac{1}{4\pi\epsilon_0} = 9\times10^9 N.$

Thus 1C is the charge which when placed at a distance of 1m from an equal and similar charge in air will repel it with force of 9×10^9 Newton

By quantization of charge

$$q = ne$$

$$n = \frac{q}{e} = \frac{1C}{1.6 \times 10^{-19} C}$$

$$n = 6.25 \times 10^{18}$$

Thus 1 C is the charge that is produced by removal of 6.25×10^{18} electron from a neutral body.

generally C is expressed in microcoulomb or nanocoulomb

$$1 \mu C = 10^{-6} C$$

$$1 nC = 10^{-9} C$$

Derive Unit of charge

The rate of flow of charge is called current

$$\text{current} = \frac{\text{charge flowing}}{\text{time}}$$

$$I = \frac{Q}{t}$$

$$\text{charge } Q = I \times t$$

$$= \text{current} \times \text{time}$$

The derived unit of charge is Ampere × second.

and dimensional formula is

$$[AT]$$

1C is the charge which when flowing in a conductor for 1 second causes a current of 1 Amp.

Other unit of charge

(i) Electrostatic unit of charge (e.s.u of charge). This is also called stat coulomb

$$1C = 3\times10^{9} \text{ stat coulomb}$$

(ii) Electromagnetic unit of charge (e.m.u. of charge)

$$1 \text{ e.m.u. of charge} = 10C = 3\times10^{10} \text{ stat coulomb}$$

Electric field strength

The region in which a charge experiences a force is called electric field.

If an infinitesimally small charge q_0 experiences a force $\vec{F}$ at a point in electric field, then the strength of field at that point is

$$\vec{E} = \frac{\vec{F}}{q_0}$$

Thus the intensity of electric field or Electric field strength at any point is equal to the ratio of the force experienced by an infinitesimally small test charge and value of charge.

If value of charge is $q_0 = +1$

then $\vec{E} = \vec{F}$

Thus intensity of electric field or electric field strength at any point is equal to the force experienced by a unit positive charge placed at that point.

or $\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$

or $\vec{E} = \vec{F}$

Dimension of electric field strength

$E = \frac{\text{force}}{\text{charge}} = \frac{[MLT^{-2}]}{[AT]}$

$= [MLT^{-3}A^{-1}]$

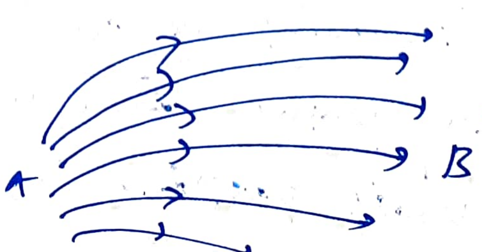
Concept of Electric field

According to Faraday each vector field may be supposed to be formed of vector line of force.

The path of motion of positive test charge, ~~pos~~ placed free in an electric field, is called electric line of force.

Thus the electric line of force in an electric field is an imaginary ~~smooth~~ smooth curve along which an isolated free positive charge (initially at rest) tends to move.

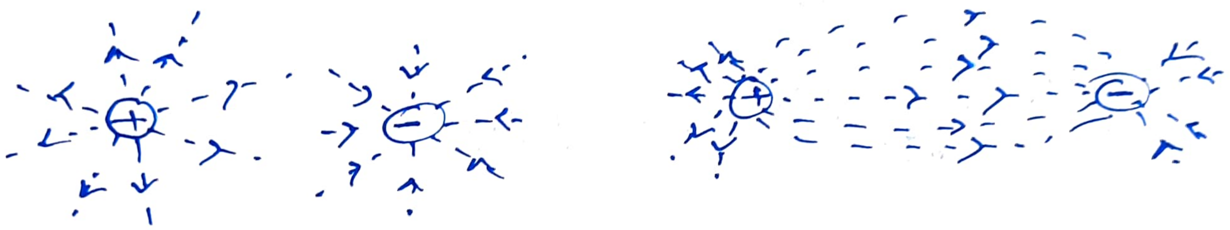
The electric field strength at any point is defined as vector quantity whose magnitude is measured by the number of electric ~~field~~ lines of force passing normally through per unit small area around that point and whose direction is along the tangent on the line of force drawn at that point.



If the electric lines of force are nearer, the electric field is strong. If the electric lines of force are far way, the electric field is weak.

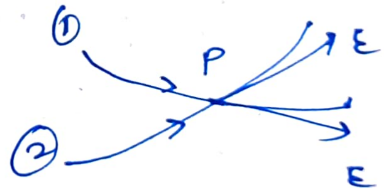
Properties of Electric lines of force

(i) The electric lines of force appear to start from positive charge and to end at negative charge or at infinity



(ii) The tangent drawn at any point on the line of force gives the direction of field.

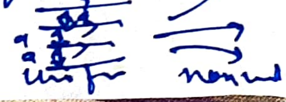
(iii) No two electric lines of force can intersect each other.



(iv) The electric lines of force are always in form of open curves. They do not form closed loops.

(v) The leelric lines of force do not pass through a conductor since $\vec{E}=0$ ~~in con.~~ inside conductor

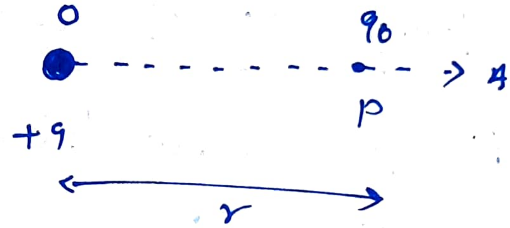
(vi) The equidistance electric lines of force represent uniform electric field and electric lines of force at different separation represents non-uniform electric fild.



Calculation of Electric field strength

(i) Electric field due to a point charge

Let a point charge q be placed at O, which produces electric field.



Let a positive test charge q_0 is placed at P at a distance r from O.

From Coulomb's Law

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q q_0}{r^2} \text{ along P to A}$$

Electric field at P

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \text{ along P to A}$$

$$\vec{E} = \frac{F}{q_0}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

where $\hat{r}$ is unit vector along P to A.

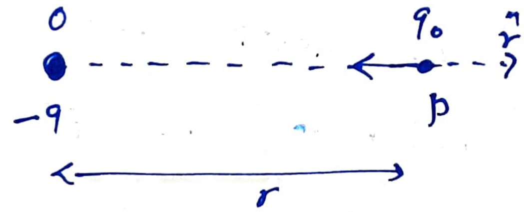
If $\vec{r}$ is position vector of point P then

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\frac{\vec{r}}{|\vec{r}|}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$$

If $-q$ charge is placed at O then electric field strength at P is.

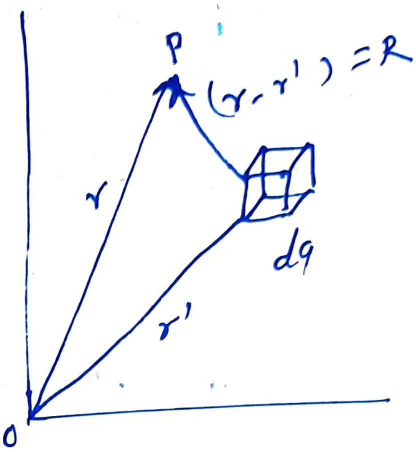


$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \text{ from P to O}$$

or $$= \vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$$

(ii) Electric field due to continuous charge distribution (5

A continuous charge distribution is supposed to be formed of a large no. of very small charge elements, let dq be a small charge element.



If $\vec{r}'$ be the position vector of charge dq w.r.to origin and position vector of point P under consideration is $\vec{r}$.

The field at P due to this charge element.

$$dE = \frac{dq}{4\pi\epsilon_0 R^2}$$

where $R = r - r'$

Therefore total electric field strength at P is

$$E = \int dE = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{R^2}$$

in vector form

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{R^3}\vec{R}$$

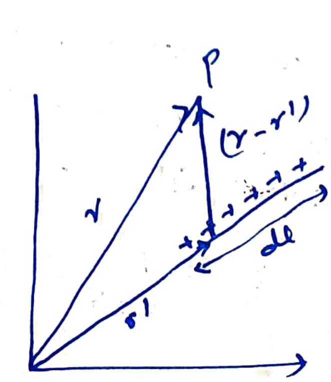
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

(a) Due to line charge

If λ is charge per unit length then charge on element of length dl is

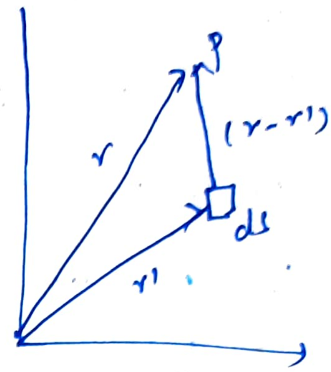
$$dq = \lambda\, dl$$

Then $E = \frac{1}{4\pi\epsilon_0}\int_L \frac{\lambda\, dl}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$



(b) Due to surface charge distribution

Let charge be distributed at the surface and surface charge density is σ, then charge on surface element of area ds is



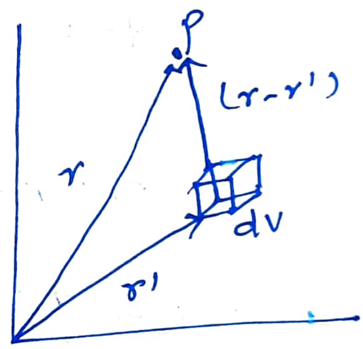
$$dq = \sigma ds$$

therefore electric field strength at P is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_s \frac{\sigma ds}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

(c) Due to volume charge distribution

Let charge be distributed ~~at the~~ over volume V of body and ρ is volume charge density, then charge in small volume element dV is



$$dq = \rho dV$$

then electric field ~~str~~ strength at P is

$$E = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho dV}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

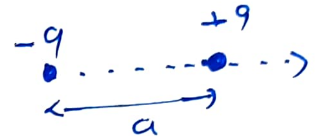
(d) General charge distribution

If the charge is distributed in space a length L, surface S and volume V and there are n point charges $q_1, q_2 \cdots q_n$ then net electric field strength at point P located at $\vec{r}$ from origin will be

$$E = \frac{1}{4\pi\epsilon_0}\left[\int_L \frac{\lambda dl}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}') + \int_s \frac{\sigma ds}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}') + \int_V \frac{\rho dV}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}') + \sum_{i=1}^{n} q_i \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]$$

Electric field strength due to electric dipole

<u>Electric dipole</u> :- It is a system of two equal and opposite charges separated by a small distance



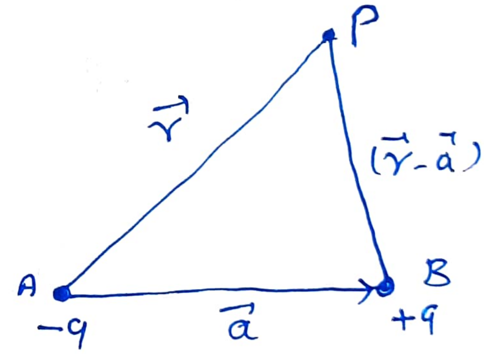
Electric dipole moment

It is a product of magnitude of either charge and separation b/w the two charges and direction directed from negative charge to positive charge.

dipole moment

$$\vec{p} = q \cdot \vec{a}$$

Electric field strength



Let P is a observation point with ~~op~~ position vector $\vec{r}$ w.r.to A and $(\vec{r}-\vec{a})$ w.r.to B.

The electric field strength at point P due to charge (-q) is

$$E_A = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$$

and due to charge (+q) is

$$E_B = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{a}|^3}(\vec{r}-\vec{a})$$

The electric field strength at P due to both charg of dipole is

$$E(r) = E_A + E_B = \frac{q}{4\pi\epsilon_0}\left[\frac{(\vec{r}-\vec{a})}{|\vec{r}-\vec{a}|^3} - \frac{r}{r^3}\right] \quad \text{--- (1)}$$

If $r >> a$ than from first term of eqn ① with binomial expression

$$|\vec{r}-\vec{a}|^{-3} = |(\vec{r}-\vec{a})^2|^{-3/2}$$

$$= [r^2 - 2\vec{r}.\vec{a} + a^2]^{-3/2}$$

$$= \frac{1}{r^3}\left[1 - \frac{2\vec{r}.\vec{a}}{r^2} + \frac{a^2}{r^2}\right]^{-3/2}$$

$$= \frac{1}{r^3}\left[1 + \frac{3}{2}\left(\frac{2\vec{r}.\vec{a}}{r^2} + \frac{a^2}{r^2}\right)\right]$$

since $r >> a$ then

$$= \frac{1}{r^3}\left[1 + \frac{3\,\vec{r}.\vec{a}}{r^2}\right]$$

from eqn ①

$$E(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^3}\left(1 + \frac{3\vec{r}.\vec{a}}{r^2}\right)(\vec{r}-\vec{a}) - \frac{\vec{r}}{r^3}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\cancel{\vec{r}/r^3} - \frac{\vec{a}}{r^3} + \frac{3(\vec{r}.\vec{a})\vec{r}}{r^5} - \frac{3(\vec{r}.\vec{a}).\vec{a}}{r^5} - \cancel{\vec{r}/r^5}\right]$$

since again $r >> a$ than

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{3(\vec{r}.\vec{a})\vec{r}}{r^5} - \frac{\vec{a}}{r^3}\right]$$

But $qa = p$, we get

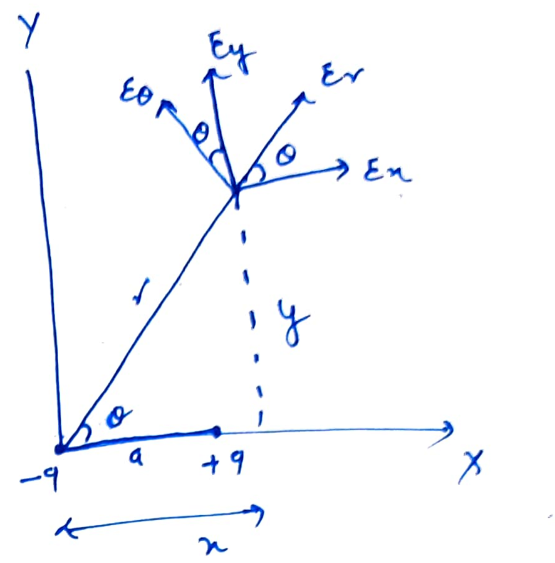
$$E(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{r}.q\vec{a})\vec{r}}{r^5} - \frac{q\vec{a}}{r^3}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{r}.\vec{p})\vec{r}}{r^5} - \frac{p}{r^3}\right] \quad \text{--- ②}$$

this is required relation for electric field strength.

Component of electric field :-

Let x axis is along the direction of p and (x, y) be the coordinates of P. then x and y components of field are.



$$\cancel{E_x = \frac{1}{4\pi\epsilon_0}\left[\frac{3\, x p\, r \cos\theta}{r^5} - \frac{p}{r^3}\right]}$$

$$= \cancel{\frac{1}{4\pi\epsilon_0}\left[\frac{3p\cos\theta}{r^3} - \frac{p}{r^3}\right]}$$

$$E_x = \frac{1}{4\pi\epsilon_0}\left[\frac{3px^2}{r^5} - \frac{p}{r^3}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3p\left(\frac{x}{r}\right)^2 - p\right]$$

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[3\cos^2\theta - 1\right) \quad -(3)$$

and $E_y = \frac{1}{4\pi\epsilon_0}\left[\frac{3xp}{r^5}\cdot y\right]$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{3p}{r^3}(x/r)(y/r)\right]$$

$$E_y = \frac{1}{4\pi\epsilon_0}\frac{3p}{r^3}\cos\theta\cdot\sin\theta \quad -(4)$$

the polar component of electric field

$$E_r = \frac{1}{4\pi\epsilon_0}\left[\frac{3(rp\cos\theta)r}{r^5} - \frac{p\cos\theta}{r^3}\right]$$

$$E_r = \frac{1}{4\pi\epsilon_0}\left[\frac{2p\cos\theta}{r^3}\right]$$

and $E_\theta = \frac{1}{4\pi\epsilon_0}\left[0 - \left(-\frac{p\sin\theta}{r^3}\right)\right]$

$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2 + y^2$
along x axis $\theta = 0$
$r.p = rp\cos 0 = rp$

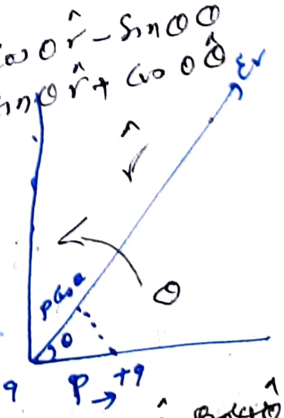
x component of P is
$P_x = P$
and y is
$P_y = 0$
x component of r is x and y component of r is y

$$E_x = \frac{1}{4\pi\epsilon_0}\left[\frac{3x.P.x}{r^5} - \frac{p}{r^3}\right]$$

$$E_r = \left[\frac{(\hat{i}x + \hat{j}y)\cdot P\hat{i}\,[x\hat{i} + \hat{j}y]}{r^5} - \frac{P\hat{i}}{r^3}\right]$$

$$E_y = \left[\frac{xP}{r^5}(x\hat{i} + y\hat{j}) - \frac{P\hat{i}}{r}\right.$$

$\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$
$\hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$



$$E = \left[\cancel{[x\cos\theta(\cos\theta\hat{r} - \sin\theta\hat{\theta})} \right.$$
$$\cancel{+ r\sin\theta(\sin\theta\hat{r} + \cos\theta\hat{\theta})}$$

$$r\cos\theta\, p\, \cancel{(\cos\theta\hat{r} - \sin\theta\hat{\theta})}.$$

$$\frac{[r\cos\theta(\cos\theta\hat{r} - \sin\theta\hat{\theta}) + r\sin\theta(\sin\theta\hat{r} + \cos\theta\hat{\theta})}{r^5} - \frac{P(\cos\theta\hat{r} - \sin\theta\hat{\theta})}{r^3}$$

$$E_\theta = +\frac{1}{4\pi\epsilon_0} \frac{p\sin\theta}{r^3}$$

That is ~~the~~ radial and transverse (azimuthal) component of electric field.

$$E_r = \frac{1}{4\pi\epsilon_0}\left(\frac{2p\cos\theta}{r^3}\right) \quad - (5)$$

$$E_\theta = +\frac{1}{4\pi\epsilon_0}\left(\frac{p\sin\theta}{r^3}\right) \quad - (6)$$

Special ~~cas~~ case

(i) If the point p ~~dipole~~ is along x axis / or along the axis of dipole then $\theta = 0$

therefore $E_r = \frac{1}{4\pi\epsilon_0}\frac{2p}{r^3}$ and $E_\theta = 0 = E_y$

$= E_x$

that is electric field is directed along x axis

(ii) If the point p lies perpendicular to x axis or axis of dipole then

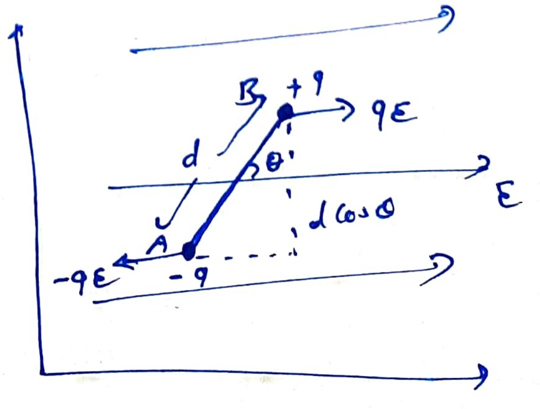
$\theta = 90$

$E_r = 0$ and $E_\theta = \frac{+p}{4\pi\epsilon_0 r^3}$

$E_x = -\frac{1}{4\pi\epsilon_0}\frac{p}{r^3}$, $E_y = 0$

Electric dipole in Uniform Electric field

Let a dipole consisting of two charges $-q$ and $+q$ separated by a distance 'd' is placed in a uniform electric field. The potential energy of dipole is



$W = \cancel{q(\phi_B - \phi_A)}\ q(V_B - V_A)$

$W = -qd\cos\theta E =$

$= -qd \cdot E$

$W = -\vec{p} \cdot \vec{E}$ — (1)

τ = force × distance b/w lines of action of force

$\tau = qE \times d\sin\theta = qdE\sin\theta$

$= pE\sin\theta$

$= \vec{p} \times \vec{E}$

The energy does not depend on the position of dipole and depends only on the angle b/w $\vec{p}$ and $\vec{E}$. therefore there is no translational force acting on the dipole.

The field exerts a force $+qE$ on the charge $+q$ and $-qE$ on $-q$, there is a couple τ acting on it

$dW = \tau d\theta$

$$\tau = -\frac{dW}{d\theta} = -pE\frac{d}{d\theta}(-\sin\theta)$$

$$W = \int_{\theta}^{90} pE\sin\theta\, d\theta = [pE\cos\theta]_{\theta}^{0} = -pE\cos\theta$$

$\tau = pE\sin\theta$

$\tau = \vec{p} \times \vec{E}$ — (2)

Which tends to turn the dipole into a position parallel to the field.

$$W = \int_{\theta_0}^{\theta_f} \tau d\theta = \int_{\theta_0}^{\theta_f} pE\sin\theta d\theta$$

$= pE(\cos\theta_0 - \cos\theta_f)$

If $\theta_0 = 90$, $\theta_1 = 0$

$= -pE\cos\theta$

$= -p \cdot E$

Case I

when $\theta = 0$

then $\tau = 0 = \tau_{min}$

i.e. dipole is aligned with the field and no torque acts on it. This is equilibrium state.

Case II

when $\theta = 90$

$\tau = PE = \tau_{max}$

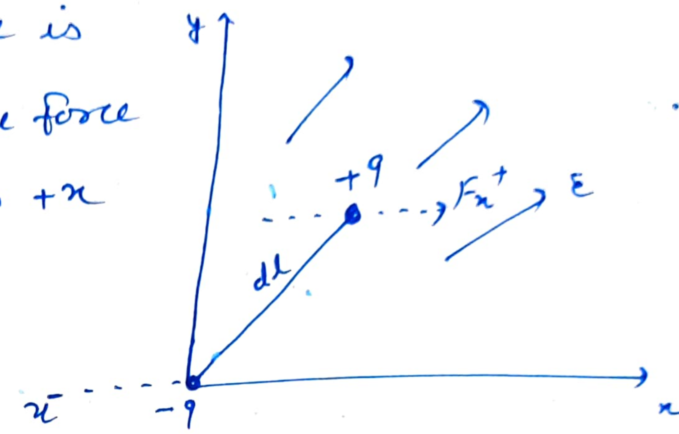
i.e. Torque acting on it is maximum.

)

Electric dipole in a non-uniform field

If the field in which the dipole is placed is non-uniform a translational force is exerted on it

Let charge $-q$ of dipole is placed on the origin. The force acting on the dipole in $+x$ direction is



$$F_x^+ = q(E_x + dE_x)$$

whereas in $-x$ direction force is

$$F_x^- = qE_x$$

The net component of force in $+x$ direction

$$F_x = q\,dE_x$$

Since the field is non-uniform

$$dE_x = \left(\frac{\partial E_x}{\partial x}\right)dx + \left(\frac{\partial E_x}{\partial y}\right)dy + \left(\frac{\partial E_x}{\partial z}\right)dz$$

$$\therefore\ F_x = q\left[dx\frac{\partial E_x}{\partial x} + dy\frac{\partial E_x}{\partial y} + dz\frac{\partial E_x}{\partial z}\right]$$

$$= q\left[dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right]E_x$$

$$= q(dl\cdot\nabla)E_x$$

$$= (p\cdot\nabla)E_x$$

Similarly the components of force on y and z direction are

$$F_y = (p\cdot\nabla)E_y$$

$$F_z = (p\cdot\nabla)E_z$$

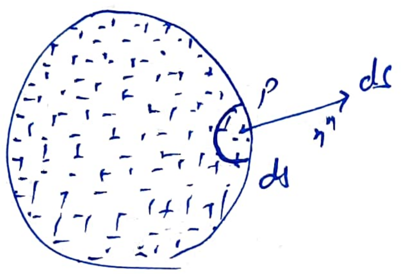
The net force acting on dipole is

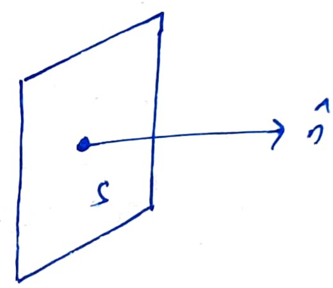
$$\vec{F} = (\vec{P}\cdot\nabla)\vec{E}$$

Necessary basis concepts for Gauss's theorem

1. Vector Area

Any surface is supposed to form a large no. of small elements. Each surface elements is a vector quantity. Let a surface element around a point P, having magnitude ds and direction along the outward drawn normal ($\hat{n}$)



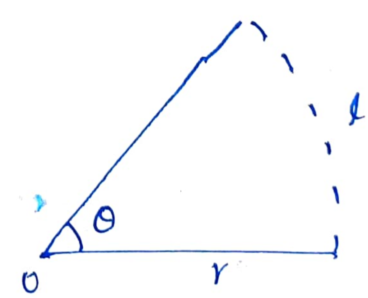


If the entire surface S is plane surface, then its magnitude is S and direction along outward drawn normal to plane of surface

$$\vec{S} = S\hat{n}$$

2. Solid Angle

The angle subtended by an arc of a circle at the centre of circle is called plane angle



$$\text{plane angle} = \frac{\text{length of arc}}{\text{radius}}$$

$$\theta = l/r$$

$l = r$, then $\theta = 1$ rad.

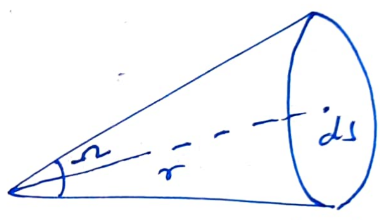
Thus 1 radian is the angle subtended by the arc of circle whose length is equal to radius

Similarly the angle subtended by an element of surface of a sphere at its centre is called the solid angle

Its unit is steradian and denoted by Ω

The angle subtended by the surface area Δs of the sphere of radius r at the centre is

$$\Omega = \frac{\Delta s}{r^2}$$



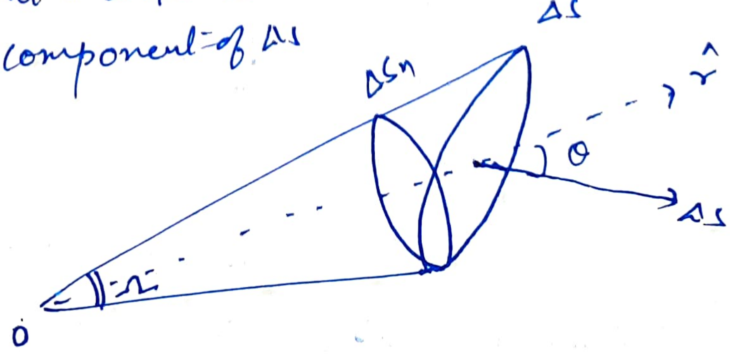
$\Delta s = r^2$
then $\Omega = 1$ steradian

Thus 1 ~~st~~ ster. is the solid angle subtended by that part of surface of sphere whose surface area is equal to the square of radius of that sphere.

In case the surface area is not spherical or point is not the centre ~~sphere~~ of sphere, then the solid angle

$$\Omega = \frac{\text{normal surface area}}{(\text{perpendicular distance of surface from the point})^2}$$

If Δs is surface area at a distance r from point O and normal component of Δs is Δs_n then



solid angle

$$\Omega = \frac{\Delta s_n}{r^2}$$

Then $\Delta s_n = \Delta s \cos\theta$

θ is angle b/w $\hat{r}$ and direction of Δs

$$\Omega = \frac{\Delta s \cos\theta}{r^2} = \frac{\Delta s \cos\theta\, r}{r^3}$$

$$\Omega = \frac{\vec{\Delta s}\cdot\vec{r}}{r^3}$$

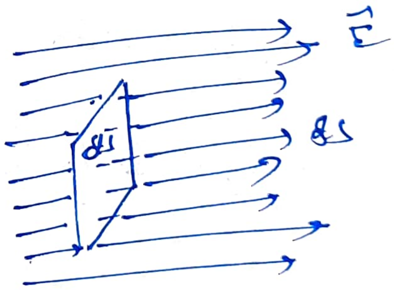
Total solid angle subtended by a close surface at any internal point is same as that of a complete surface of sphere at the centre of sphere

$$\Sigma\Omega = \frac{4\pi r^2}{r^2} = 4\pi$$

That is a solid angle subtended by a closed surface at internal point is always 4π.

Electric flux

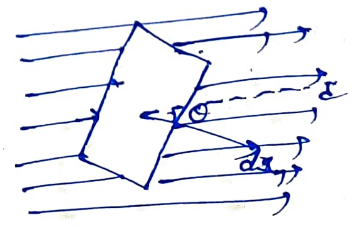
Total no. of electric lines of force passing through ~~that~~ the given surface normally.



Since electric field strength is no. of lines of force passing per unit area normally.

If δs is elementary area normal to the $\vec{E}$ line electric flux through surface area $\delta\phi = E\,\delta s$

If δs is not perpendicular to E ~~the~~ but whose normal makes an angle θ with E, the normal component of surface is



$$\delta s_n = \delta s \cos\theta$$

The electric flux through surface area δs is

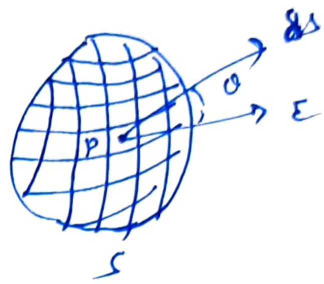
$$\delta\phi = E\,\delta s_n = E(\delta s\cos\theta)$$

$$\delta\phi = \vec{E}\cdot\vec{\delta s}$$

The electric flux through whole surface S

If the element of surface are same

$$\phi = \Sigma E \cdot \delta s$$



If the surface is continuous and E is different at different surface element then electric flux is

$$\phi = \int_S E \cdot ds$$

Gauss's theorem

K. F. Gauss gave a theorem, known as Gauss's theorem.

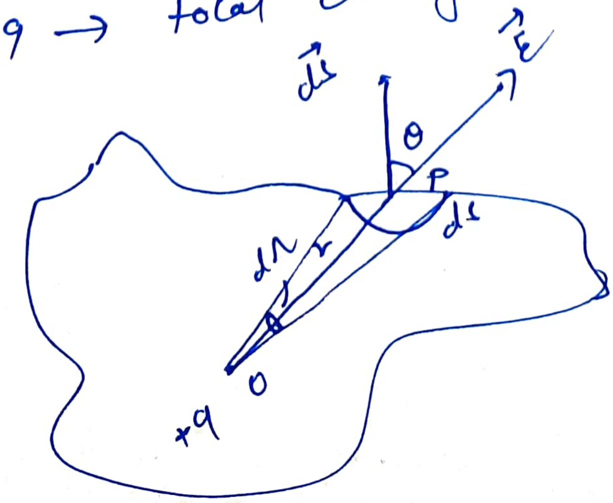
The net outward normal electric flux through any closed surface of any shape (called Gaussian surface) is equal $1/\epsilon_0$ times the total charge contained with-in that surface.

i.e. $$\oint_S E \cdot ds = \frac{1}{\epsilon_0} \Sigma q$$

where $\oint_S \rightarrow$ surface integral over whole of the closed surface.

$\Sigma q \rightarrow$ total charge enclosed by surface S.

Proof



Let q charge is placed at O with in closed surface.
E be electric field at P and OP = r.
ds is small area of the surface at P
then electric field through ds is

$$d\phi = E.ds$$

Field at P is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\vec{r}$$

since $\vec{r} = r\hat{r}$

$$d\phi = E.ds = \frac{1}{4\pi\epsilon_0} q . \frac{\vec{r}.\vec{ds}}{r^3}$$

$$= \frac{q}{4\pi\epsilon_0} d\Omega$$

where $d\Omega = \frac{\vec{r}.\vec{ds}}{r^3} = \frac{ds\cos\theta}{r^2}$ = solid angle substended by area ds at point O and θ is angle b/w $\vec{ds}$ and $\vec{E}$.

Net electric flux through whole of closed surface

$$\phi = \oint_s \vec{E}.\vec{ds} = \frac{q}{4\pi\epsilon_0}\oint d\Omega$$

since $\oint d\Omega = 4\pi$ then

$$\phi = \frac{q}{4\pi\epsilon_0} . 4\pi = q/\epsilon_0$$

$$\therefore \phi = 1/\epsilon_0 . q$$

If there are large no. of charges $q_1, q_2 \ldots q_n$ inside the closed surface then electric flux is

$$\phi = \frac{1}{\epsilon_0}q_1 + \frac{1}{\epsilon_0}q_2 + \frac{1}{\epsilon_0}q_3 \ldots + \frac{1}{\epsilon_0}q_n$$

$$\phi = \frac{1}{\epsilon_0}\Sigma q,$$ where Σq is algebric sum of charges.

If the charge is uniformly distributed, then Gauss's theorem

$$\oint \vec{E}\cdot\vec{ds} = \frac{1}{\epsilon_0}\int_v \rho \cdot dv$$

$\left\{ \begin{array}{l} \text{charge density} = \rho = \frac{dq}{dv} \\ dq = \rho dv \\ q = \int dq = \int_v \rho dv \end{array} \right.$

where ρ is charge density and dv is small element of volume and volume integral is taken through the volume bounded by surface s.

Application of Gauss's theorem

It provides a convenient method for determination of electric field strength in symmetrical cases. Main three steps are involved in this method.

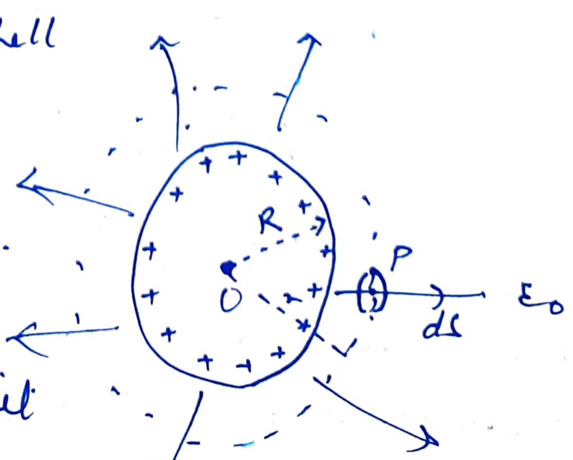
(i) To image a Gaussian surface symmetrical to given charge.

(ii) Computing electric flux through this surface.

(iii) To equate this flux to $1/\epsilon_0 \times$ charge enclosed by this surface.

① Electric field strength due to uniformly charged thin spherical shell

(i) Electric field outside the shell



let R be the radius of spherical shell centre at O.

let $+q$ charge is uniformly distributed on it

r is the radius of Gaussian surface and E_0 is field on Gaussian surface.

Electric flux through the Gaussian surface

$$\phi = \oint \vec{E}\cdot d\vec{s}$$

$$= \int E_o \, ds \cos 0 = \int E_o \, ds$$

$$\phi = E_o \int ds = E_o \, 4\pi r^2$$

$$\phi = \frac{1}{\epsilon_0} \times \text{charge enclosed}$$

$$E_o \, 4\pi r^2 = \frac{1}{\epsilon_0} q$$

$$E_o = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad \text{for } r > R$$

thus for external points the uniformly charged spherical shell behaves as if the whole charge is concentrated at the centre of the shell.

(ii) On surface of shell

for points on surface of shell

$$r = R$$

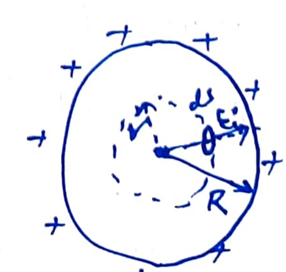
then

$$E_s = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2}$$

(iii) Field inside the shell

let field inside the shell is E_i

Consider a Gaussian surface of radius is r then electric flux through Gaussian surface



$$\phi = \oint_s E_i \cdot ds$$

$$= \int E_i \, ds \cos 0 = E_i \times 4\pi r^2$$

From Gaussian theorem

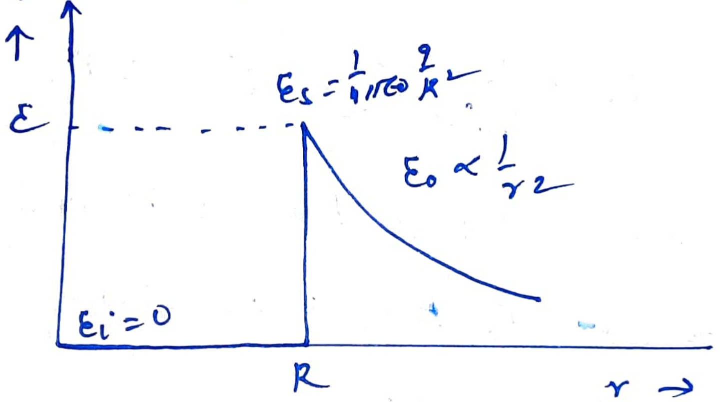
$$\oint \vec{E_i} \cdot \vec{ds} = \frac{1}{\epsilon_0} \text{ charge enclosed}$$

$$= \frac{1}{\epsilon_0} \times 0$$

$$E_i = 0$$

Thus electric field at each point inside a charged thin spherical shell is zero.

variation of electric field with distance



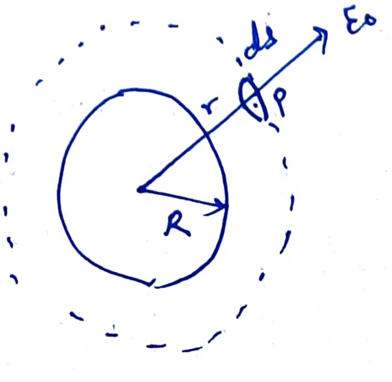
② <u>Electric field due to uniformly charged non-conducting sphere.</u>

Let Q charge is uniformly distributed on the sphere of radius R. The charge density

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

i) Field outside the sphere

Let r be the radius of Gaussian surface. Electric field at P is E_0 and ds is small surface element.



The total electric flux through the whole surface.

$$\phi = \oint_S \vec{E_0} \cdot \vec{ds} = \int E_0\, ds \cos 0$$

$$= E_0\, 4\pi r^2$$

From Gauss theorem

$$\phi = E_0\, 4\pi r^2 = \frac{1}{\epsilon_0} \text{ charge enclosed}$$

$$= \frac{1}{\epsilon_0} Q$$

$$E_0 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

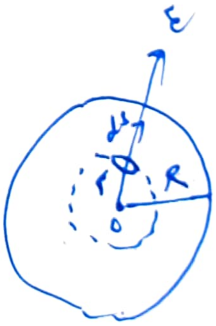
(ii) Field at the surface of Sphere

$r = R$

then

$$E_s = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2}$$

(iii) Field inside the sphere



Let r be the radius of Gaussian surface inside the sphere then total flux through Gaussian surface

$$\phi = \oint E_i \cdot ds = \int E_i\, ds \cos 0$$

$$= E_i \int ds = E_i\, 4\pi r^2$$

$$= E_i\, 4\pi r^2$$

the charge enclosed by Gaussian surface

$$q = \rho \frac{4}{3}\pi r^3$$

From Gauss theorem

$$E_i \cdot 4\pi r^2 = \frac{1}{\epsilon_0} q = \frac{1}{\epsilon_0} \rho \frac{4}{3}\pi r^3$$

$$= \frac{1}{\epsilon_0} \frac{Q}{4/3\, \pi R^3} \frac{4}{3}\pi r^3$$

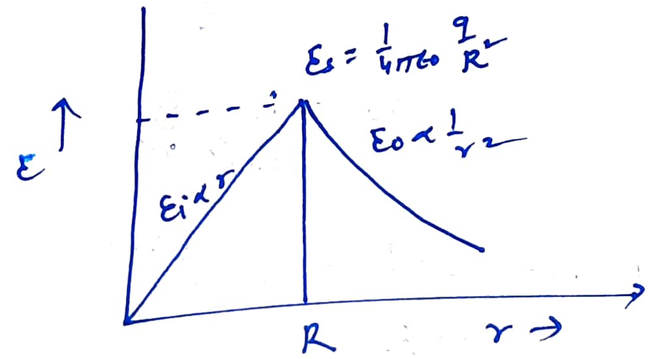
$$E_i = \frac{1}{\epsilon_0} \frac{Q r^3}{R^3} \cdot \frac{1}{4\pi r^2}$$

$$E_i = \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3}$$

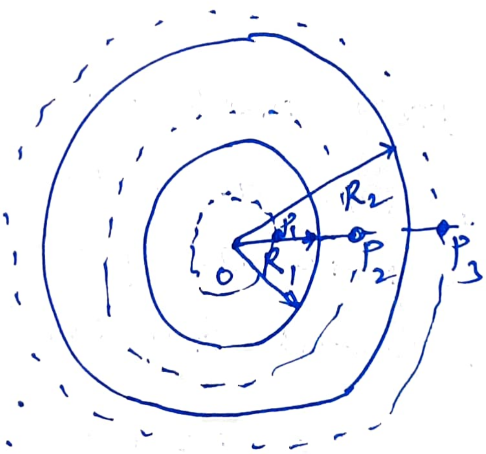
$E_i \propto r$

$E_i \propto \frac{1}{R^3}$

variation of field with distance



(3) Electric field due to cocentric spherical shell



Let there are two cocentric spherical shell of radii R_1 and R_2 ($R_1 < R_2$) and bearing charge Q_1 and Q_2 respectively

(i) Field inside the inner shell

Let r ($r < R_1$) be the radius of Gaussian surface

From Gaussian theorem

$\phi = 1/\epsilon_0$ charge inside the shell of radius r

$E_i \cdot 4\pi r^2 = \phi = 1/\epsilon_0 \times 0$

$E_i = 0$ if $r < R_1$

(ii) Field outside both shell

Let r be the radius of Gaussian surface of radius ($r > R_2$)

Total flux = $\frac{1}{\epsilon_0}$ charge enclosed by Gaussian surface

$$E_0 \cdot 4\pi r^2 = \frac{1}{\epsilon_0}(Q_1 + Q_2)$$

$$E_0 = \frac{1}{4\pi\epsilon_0} \frac{Q_1+Q_2}{r^2}$$

(c) Field b/w shell

Let Gaussian surface lies b/w spherical shells. The radius of surface is r ($R_1 < r < R_2$)

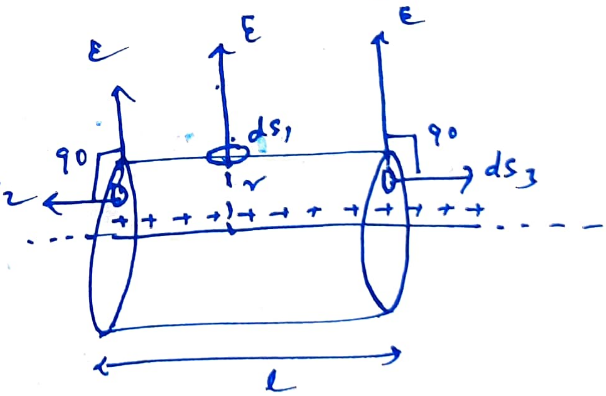
Total flux = $\frac{1}{\epsilon_0}$ charge enclosed by surface

$$E_1 \, 4\pi r^2 = \frac{1}{\epsilon_0} Q_1$$

$$E_1 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

(4) Electric field due to an infinitely long straight wire

Let λ be the linear charge density of infinitely long charge. Consider the cylindrical Gaussian surface of radius r and length l coaxial with line charge.



Consider the 3 surface elements ds_1, ds_2 and ds_3. $\vec{ds_2}$ and $\vec{ds_3}$ are parallel to wire and $\vec{ds_1}$ is normal to wire.

Total electric flux through the cylindrical Gaussian surface

$$\oint_S \vec{E} \cdot \vec{ds} = \oint_{S_1} \vec{E} \cdot \vec{ds_1} + \oint_{S_2} \vec{E} \cdot \vec{ds_2} + \oint_{S_3} E \cdot ds_3 \cos 90$$

$$= \oint_{S_1} E \, ds_1 \cos 0 + \oint_{S_2} E \, ds_2 \cos 90 + \oint_{S_3} E \, ds_3 \cos 90$$

$$= \oint_{S_1} E \, ds_1 + 0 + 0$$

$= E \int ds_1$ (area of ~~cylinder~~ curve surface)

$= E\, 2\pi r l$

By the Gaussian surface

$$\oint \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon_0} \text{ charge enclosed}$$

$$= \frac{1}{\epsilon_0} (\lambda l)$$

$$E\, 2\pi r l = \frac{1}{\epsilon_0} \lambda l$$

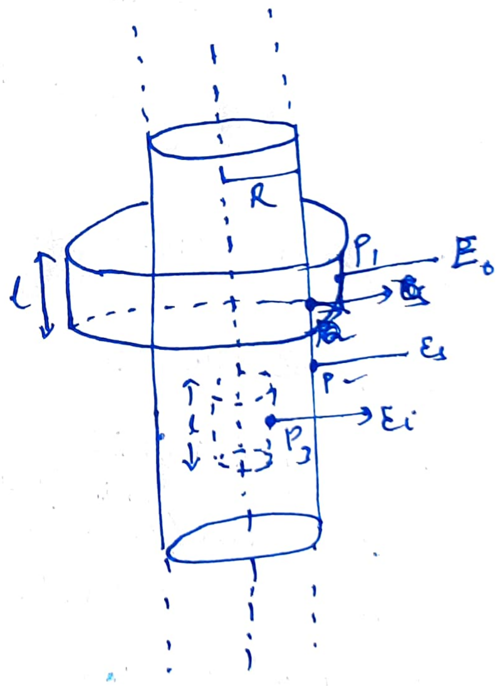
$$E = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{r}$$

(5) Field due to a uniform infinite cylindrical charge

Consider the charge is uniformly distributed within an infinite cylinder of radius R

If λ is linear charge density and ρ is volume charge density

then charge on the cylinder of radius R and length l is



$$\pi R^2 l \rho = \lambda l$$

$$\rho = \frac{\lambda}{\pi R^2}$$

(i) Field outside the cylinder

Field outside the charge distribution $r > R$

when point P_1 lies outside the charge distribution, let field at point is E_0 then flux through the Gaussian surface

$$\phi = \oint_S E_0\, ds = E_0 \oint_S ds$$

$$= E_0\, 2\pi r l$$

from Gauss's theorem

$\phi = \oint E \cdot ds = \frac{1}{\epsilon_0} \times$ total charge enclosed by Gaussian surface.

$$E_0 \cdot 2\pi r l = \frac{1}{\epsilon_0} \lambda l$$

$$E_0 = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{r}$$

(ii) Field on the surface of charge distribution

let P_1 be the point on the surface of charge distribution then from Gauss's theorem

$$\oint E_1 \cdot ds = \frac{1}{\epsilon_0} \times \text{total charge enclosed by surface}$$

$$E_1 (2\pi R l) = \frac{1}{\epsilon_0} (\lambda l)$$

$$E_1 = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{R}$$

(iii) Field inside the charge distribution

let P_2 be the point at a distance $r (< R)$ from the axis of cylinder. the field at P_2 is E_i then from Gauss's theorem

$$2\pi r l E_i = \frac{1}{\epsilon_0} \text{ charge enclosed by Gaussian surface}$$

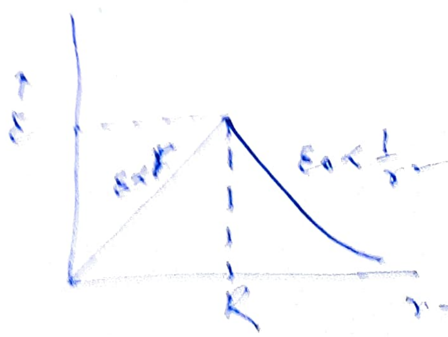
$$2\pi r l E_i = \frac{1}{\epsilon_0} (\pi r^2 l \rho)$$

$$E_i = \frac{1}{\epsilon_0} \frac{1}{2\pi r l} \pi r^2 l \cdot \frac{\lambda}{\pi R^2}$$

$$E_i = \frac{1}{4\pi\epsilon_0} \frac{2\lambda r}{R^2}$$

$E_i \propto r$

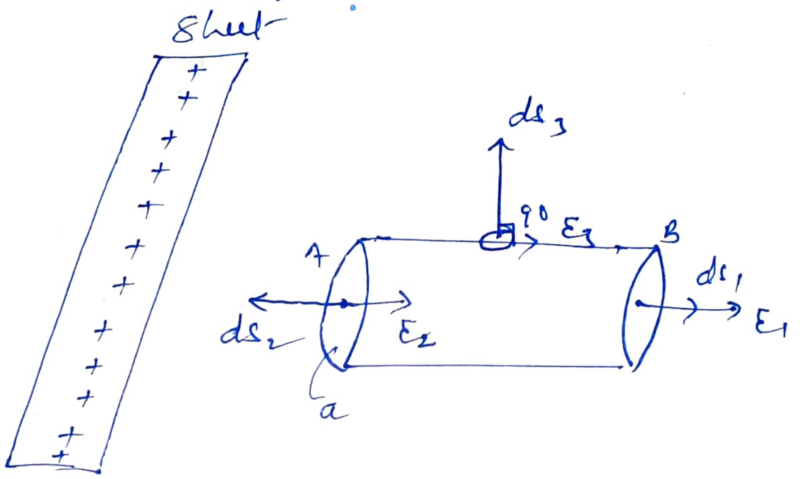
$E_i \propto \frac{1}{R^2}$



6) Electric field strength due to an infinite non-conducting plane sheet of charge

Let charge be uniformly distributed over the surface of a thin non-conducting infinite sheet.

Let σ be the surface charge density



Consider a cylindrical Gaussian surface on one side of sheet bounded by two flat surfaces A and B each of area a parallel to sheet and cylindrical surface perpendicular to sheet

Let S_1, S_2 be flat faces and S_3 be curve face of cylinder

Total flux through the Gaussian cylinder

$$\oint E \cdot ds = \oint_{S_1} E_1 \cdot ds_1 + \oint_{S_2} E_2 \cdot ds_2 + \oint_{S_3} E_3 \cdot ds_3$$

$$= \oint_{S_1} E_1 ds_1 \cos 0 + \oint_{S_2} E_2 ds_2 \cos 180 + \int_{S_3} E_3 ds_3 \cos 90$$

$$= E_1 \int ds_1 - E_2 \int ds_2$$

$$= E_1 a - E_2 a$$

Total flux $= (E_1 - E_2) a$

charge enclosed by Gaussian surface $= 0$

then from Gauss's theorem

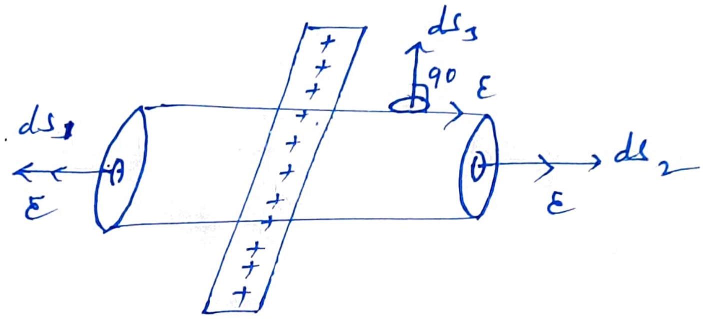
Total flux $= \phi = (E_1 - E_2) a = \frac{1}{\epsilon_0}$ charge enclosed

$$(E_1 - E_2)\, a = \frac{1}{\epsilon_0} \times 0 = 0$$

$$E_1 = E_2$$

i.e field strength is same at all point near the sheet

For determine the E near the sheet, consider the cylindrical Gaussian surface bounded by two plane faces A and B lying on the opposite sides and parallel to sheet. and cylindrical surface normal to the sheet



Total flux through the Gaussian cylinder

$$\oint \vec{E}.\vec{ds} = \oint_{S_1} \vec{E}.\vec{ds_1} + \oint_{S_2} \vec{E}.\vec{ds_2} + \oint_{S_3} \vec{E}.\vec{ds_3}$$

$$= \oint_{S_1} E\, ds_1 \cos 0 + \oint_{S_2} E\, ds_2 \cos 0 + \oint_{S_3} E\, ds_3 \cos 90$$

$$= E\int ds_1 + E\int ds_2$$

$$= Ea + Ea$$

$$\phi = 2Ea$$

~~The change~~ From Gauss's theorem

$$\oint \vec{E}.\vec{ds} = \frac{1}{\epsilon_0} \text{ charge enclosed by Gaussian cylinder surface}$$

$$= \frac{1}{\epsilon_0} \sigma a$$

$$2Ea = \frac{\sigma a}{\epsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}$$

$\vec{E}$ is independent of distance

~~the~~ Electric Potential

The electric field produced by a charge may be described in two ways

(i) by Electric field strength

(ii) by Electric potential

Electric potential :-

The electric potential at any point in electric field is defined as the work done by external force in ~~carr~~ carrying unit positive test charge from infinity to that point, without any acceleration

The potential at infinite is assumed zero.

If W is workdone in bringing positive test charge q_0 from infinity to any point in electric field, then potential at that point

$$\phi \text{ or } V \phi = \frac{W}{q_0}$$

It is scalar quantity and SI unit of it is Joule/coulomb or volt.

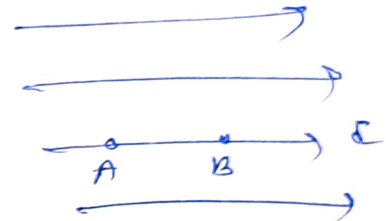
$$1 \text{ volt} = \frac{1 \text{ Joule}}{1 \text{ coulomb}}$$

1 volt is the electric potential at a point in an electric field if the work done in bringing one coulomb of charge from infinity to that point is 1 Joule, provided that charge of 1 coulomb does not affect the original electric field.

potential difference :-

The work done by external force in carrying per unit positive test charge from one point to another point in an electric field is called the electric potential difference b/w those two points.

If ~~WAB~~ W_{BA} is work done in carrying positive test charge q_0 from point B to A against the direction of electric field, the potential difference b/w A and B is



$$\phi_A - \phi_B = \frac{W_{BA}}{q_0}$$

unit of potential difference is also volt.

Therefore one volt is the potential difference b/w two points in an electric field if work done in carrying 1 coulomb of charge from one point to another point is 1 Joule.

physical significance

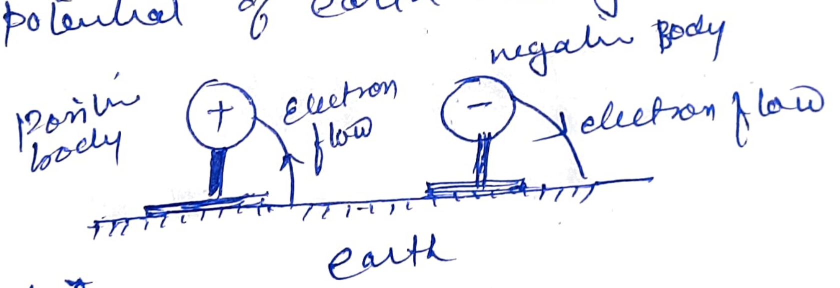
The electric potential is that physical quantity which determines the direction of flow of charge (positive). The positive charge always flows from higher potential to lower potential. There is no relation of direction of flow of charge with quantity of charge.

The negative charge always flows from ~~higher~~ lower potential to ~~lower~~ higher potential

When two conductors are placed in contact, the electrons flow from lower potential to higher potential until their potentials become equal.

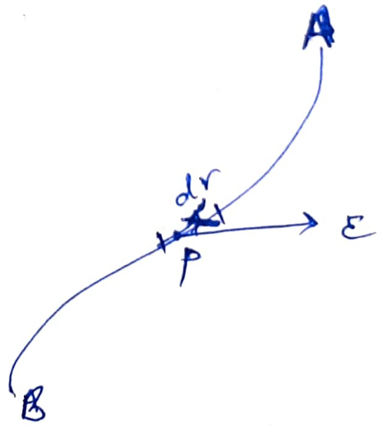
The earth as a Reference of Potential

The earth is taken as reference point for zero potential. The earth is good conductor and its size is so large that any charge given to or taken from earth does not at all alter its electric condition. So that potential of earth always remain the same.



Electric potential as linear integral of electric field

The electric potential may be defined as negative of line integral of electric field.



Let the positive charge (test) q_0 be displaced from point A to B in electric field (opposite). The field at point P is E

then force on test charge $= -q_0 E$

The work done in displacing the test charge through the small displacement dr will be

$$dW = -q_0 \vec{E} \cdot \vec{dr}$$

Total work done in displacing the charge from A to B

$$W_{AB} = -q_0 \int_A^B \vec{E} \cdot \vec{dr}$$

∴ Potential difference b/w A and B

$$\phi_B - \phi_A = \frac{W_{AB}}{q_0} = -\int_A^B \vec{E} \cdot \vec{dr}$$

If A is taken at infinity at this point $\phi_A = 0$

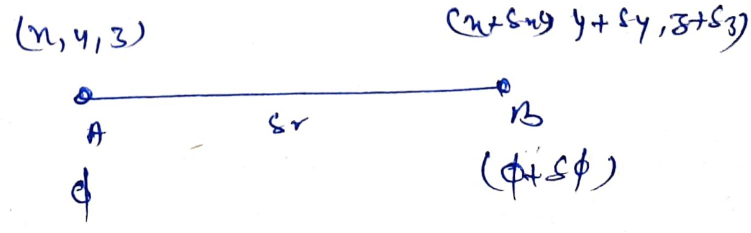
then potential at B is

$$\phi_B = -\int_{\infty}^B \vec{E} \cdot \vec{dr}$$

Thus the electric potential at any point in an electric field is defined as the negative of line integral of electric field from infinity to a given point.

<u>Electric field as negative gradient of potential</u>

Let potential at point A and B are ϕ and $\phi + \delta\phi$ and coordinates of A and B are (x, y, z) and $(x+\delta x, y+\delta y, z+\delta z)$



The potential is a function of (x, y, z) i.e.

$$\phi = \phi(x, y, z)$$

then potential difference b/w A and B is

$$\delta\phi = \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot (\hat{i}\delta x + \hat{j}\delta y + \hat{k}\delta z)$$

$$= \nabla\phi \cdot dr \quad \text{---} (1)$$

If E is electric field strength in region of point A and B

then by the definition of potential difference b/w A and B

$$\delta\phi = -\vec{E} \cdot d\vec{r} \quad \text{---} (2)$$

from eqn (1) and (2)

$$-\vec{E} \cdot d\vec{r} = \nabla\phi \cdot dr$$

$$(\vec{E} + \nabla\phi) \cdot d\vec{r} = 0$$

Since $d\vec{r} \neq 0$

Then $\vec{E} + \nabla\phi = 0$

$$E = -\nabla\phi = -\text{grad}\,\phi$$

Thus the electric field at any point is equal to the negative gradient of the potential at that point.

In terms of component

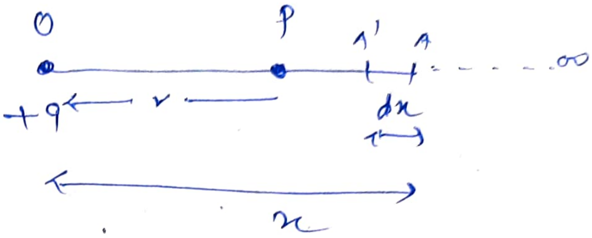
$$\vec{E} = (\hat{i}E_x + \hat{j}E_y + \hat{k}E_z) = -\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

Then $E_x = -\frac{\partial \phi}{\partial x}$, $E_y = -\frac{\partial \phi}{\partial y}$, $E_z = -\frac{\partial \phi}{\partial z}$

again $\delta\phi = -\vec{E} \cdot d\vec{r} = (\text{grad}\,\phi) \cdot dr$

Calculation of Electric potential

(i) Potential due to point charge



Let a point charge +q is placed at O. r is distance b/w OP

Let an test unit charge q_0 is brought from infinity to point P

when test charge is at A at distance x from O then

force on test charge

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{x^2}$$

The force is repulsive, carry away charge q_0 from q

therefore external force on q_0

$$F_{ext} = -\frac{1}{4\pi\epsilon_0}\frac{qq_0}{x^2}$$

The work done by this force in displacing q_0 a small distance dx is

$$dW = -\frac{1}{4\pi\epsilon_0}\frac{qq_0\,dx}{x^2}$$

The total work done in bringing charge q_0 from infinity to a point P is

$$W = -\int_{\infty}^{r}\frac{1}{4\pi\epsilon_0}\frac{qq_0\,dx}{x^2}$$

$$= -\frac{1}{4\pi\epsilon_0}qq_0\int_{\infty}^{r}\frac{1}{x^2}dx$$

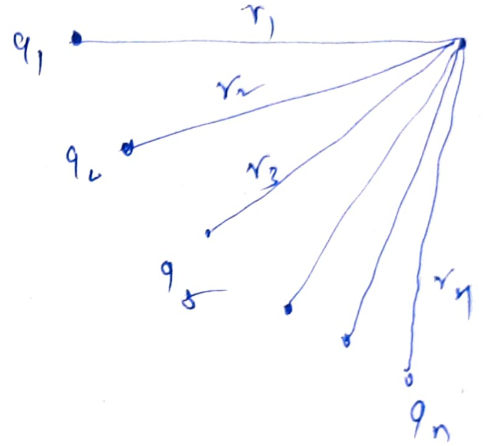
$$= -\frac{1}{4\pi\epsilon_0}qq_0\left[-\frac{1}{x}\right]_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$$

Electric Potential

$$\phi = \frac{W}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

(ii) Electric potential due to a system of point charge

The electric potential is the scalar quantity, therefore the potential is sum of potential due to all point charge



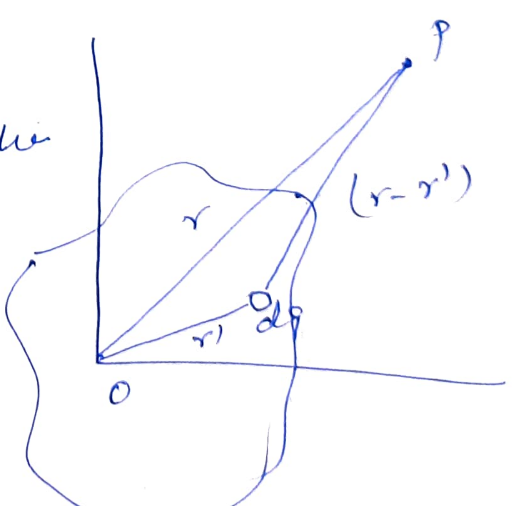
The potential at point P is

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} + \cdots \frac{1}{4\pi\epsilon_0}\frac{q_n}{r_n}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots \frac{q_n}{r_n}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\sum \frac{q_i}{r_i}$$

(iii) The electric potential due to a continuous charge distribution



Let continuous charge distribution is formed of a large no. of small charge elements

Consider a charge element dq having position vector r' w.r.t. to O

Let r be the position vector of point P relative to O.

The potential at P due to charge element dq is

$$d\phi = \frac{1}{4\pi\epsilon_0}\frac{dq}{|r - r'|}$$

The net potential due to whole charge distribution

$$\phi = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|r-r'|}$$

(a) If charge distribution is linear and λ is charge per unit length then potential at p is

$$\phi = \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda\, dl}{|r-r'|} \qquad \text{since } dq = \lambda\, dl$$

(b) If charge distribution is surface charge and σ is surface charge density

$dq = \sigma ds$

then potential is

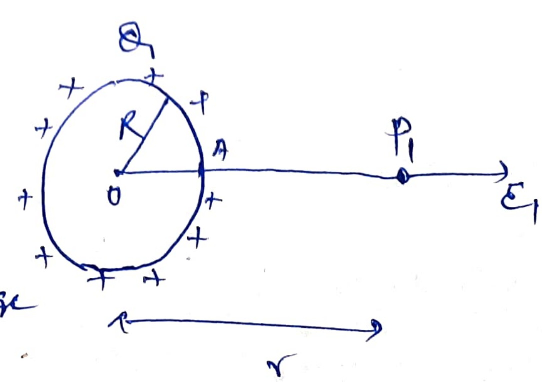
$$\phi = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma\, ds}{|r-r'|}$$

(c) If charge is distribution is volume and ρ is volume charge density then potential

$$\phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho\, dv}{|r-r'|}$$

<u>Potential due to a uniformly charge Hollow conducting sphere</u>

Let the radius of hollow sphere is R and charge on sphere is Q



~~Lt~~ let the charge distribution is uniform ~~and σ is surface charge density~~

(i) At external point

field at point P_1 is

$$E_1 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

then electric potential

$$\phi_0 = -\int_{\infty}^{r} E.dr = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}.dr$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} Q . \frac{1}{r^2} dr$$

$$= -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^{r}$$

$$\phi_0 = \frac{Q}{4\pi\epsilon_0 . r}$$

$$\phi_0 \propto \frac{1}{r}$$

(ii) At the surface of conductor

$r = R$

$$\phi_s = \frac{1}{4\pi\epsilon_0} \frac{Q}{R}$$

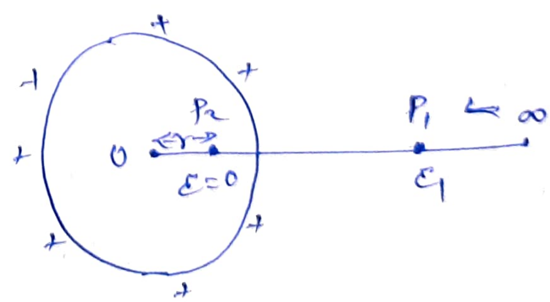
(iii) At internal point :

let P be the internal point at a

distance r ($r < R$) from centre

for external point field is

$$E_0 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$



for internal point field is

$E_1 = 0$

the potential at P_2

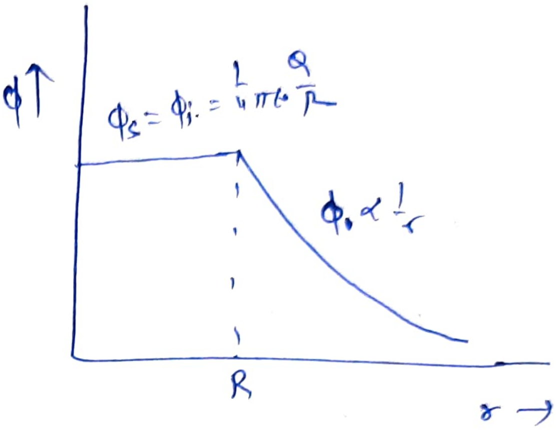
$$\phi_i = -\int_{\infty}^{r} E.dr = -\left[\int_{\infty}^{R} E_1 dr + \int_{R}^{r} E_2 . dr \right]$$

$$= -\int_{\infty}^{R} E_1 dr$$

$$\phi_i = -\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} dr$$

$$= -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r}\right]_{\infty}^{R}$$

$$\phi_i = \frac{1}{4\pi\epsilon_0} \frac{Q}{R}$$

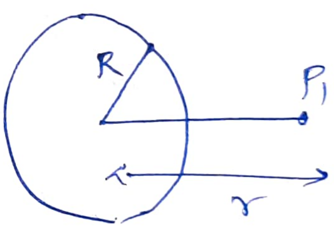


Electric potential due to uniformly charge non-conducting sphere

Let charge is distributed uniformly on sphere of radius R.

Then volume charge density

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$



(a) Outside the sphere

At P is outside the sphere at distance r ($r > R$)

The field at P

$$E_0 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

Potential at P:

$$\phi_0 = -\int_{\infty}^{r} \vec{E}_0 \cdot d\vec{r} = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} \cdot d\vec{r}$$

$$= -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{1}{r^2} dr = -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r}\right]_{\infty}^{r}$$

$$\phi_0 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

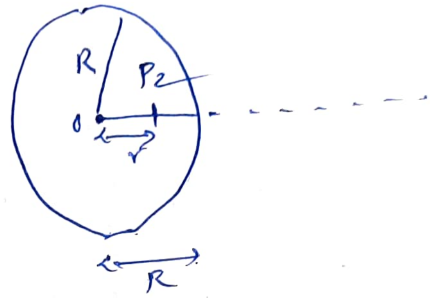
(b) At the surface

$r = R$ then

$$\phi_s = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$

(c) at inside the sphere.

Let point P_2 is inside the sphere at distance r $(r < R)$



The field at $r > R$ and $r < R$ is

$$E_o = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \quad \text{and} \quad E_{in} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{r}$$

The Potential at P_2 is

$$\phi = -\int_{\infty}^{r}\vec{E}\cdot d\vec{r} = -\left[\int_{+\infty}^{R}\vec{E_o}\cdot d\vec{r} + \int_{R}^{r}\vec{E_{in}}\cdot d\vec{r}\right]$$

$$= -\left[\int_{+\infty}^{R}\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}\cdot d\vec{r} + \int_{R}^{r}\frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{r}\cdot d\vec{r}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}Q\left[\int_{\infty}^{R}\frac{1}{r^2}dr + \int_{R}^{r}\frac{r}{R^3}dr\right]$$

$$= -\frac{Q}{4\pi\epsilon_0}\left\{\left[-\frac{1}{r}\right]_{\infty}^{R} + \frac{1}{R^3}\left(+\frac{r^2}{2}\right)_{R}^{r}\right\}$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{R} + \left(\frac{r^2}{2} - \frac{R^2}{2}\right)\frac{1}{R^3}\right]$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{R} + \frac{r^2}{2R^3} - \frac{1}{2R}\right]$$

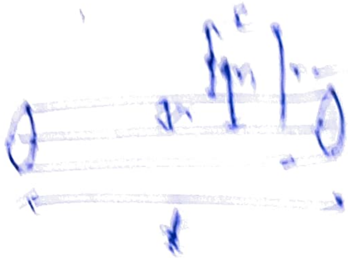
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{3}{2R} - \frac{r^2}{2R^3}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{3R^2 - r^2}{2R^3}\right]$$

(44)

Potential due to infinitely long charge wire

we know that the electric field due to infinitely long charge wire is given by



$$E = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{r}$$

where λ is linear charge density

the potential at a distance r from axis is

$$\phi_r = -\int_{r_{ref}}^{r} \vec{E}\cdot\vec{dr}$$

where r_{ref} denotes the reference distance for zero potential

Here reference distance can not be taken as infinity since the wire itself extends to infinity

therefore potential difference b/w two points at distance r_1 and r_2 from wire

the potential difference

$$\Delta\phi = -\int_{r_2}^{r_1} \vec{E}\cdot\vec{dr} = -\int_{r_2}^{r_1} \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{r} dr$$

$$= -\frac{1}{4\pi\epsilon_0} 2\lambda \left[\log r\right]_{r_2}^{r_1}$$

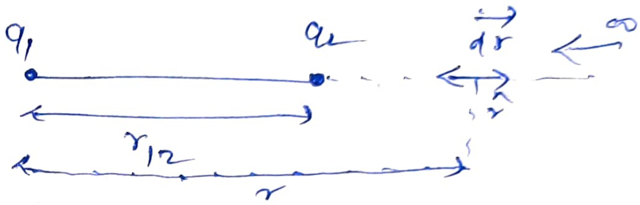
$$= \frac{1}{2\pi\epsilon_0} \lambda \log_e \frac{r_2}{r_1}$$

Electric Potential energy of a system of point charges

The electric potential energy of a system of point charge is defined as the work done in assembling this system of charges by bringing them from an infinite distance.

(i) System of two point charge

Consider two point charges q_1 and q_2. If q_1 is kept stationary, the work is required to bring the charge q_2 towards q_1.



If at any instant the distance of charge q_2 from q_1 is r then force acting b/w q_1 and q_2 is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

If charge q_2 is displaced by dr towards q_1 then work done

$$dW_1 = -\vec{F_{12}}\cdot\vec{dr} = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}\cdot dr$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}dr$$

Therefore the work done in bringing the charge q_2 from infinity to a distance r_{12} from q_1 is

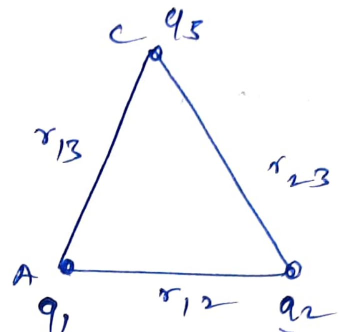
$$W_1 = -\int_{\infty}^{r_{12}}\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}dr = \frac{q_1 q_2}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{r_{12}}$$

$$W_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

(ii) System of three point charge

Let q_3 is brought from infinity to at point C whose distance from charges q_1 and q_2 are r_{13} and r_{23} resp.



The work required to do so against the electrostatic forces is

$$W_2 = -\int_{\infty}^{C} (F_{13} + F_{23}) \cdot dr$$

F_{13} force b/w q_1 and q_3

F_{23} force b/w q_2 and q_3

$$W_2 = -\int_{\infty}^{c} \vec{F}_{13} \cdot d\vec{r} - \int_{\infty}^{C} F_{23}\, dr$$

$$= \frac{1}{4\pi\epsilon_0} \int_{\infty}^{r_{13}} \frac{q_1 q_2}{r^2} dr - \frac{1}{4\pi\epsilon_0} \int_{\infty}^{r_{23}} \frac{q_2 q_3}{r^2} dr$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}}$$

Thus total work done to assemble this arrangement of 3 charges initially at infinity distance from one-another

$$W = U = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}} + \frac{1}{4\pi\epsilon_0} \frac{q_3 q_1}{r_{13}} = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$$

(iii) System of n charges

then Potential energy

$$U = \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}} + \dots \frac{q_i q_j}{r_{ij}} \right]$$

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

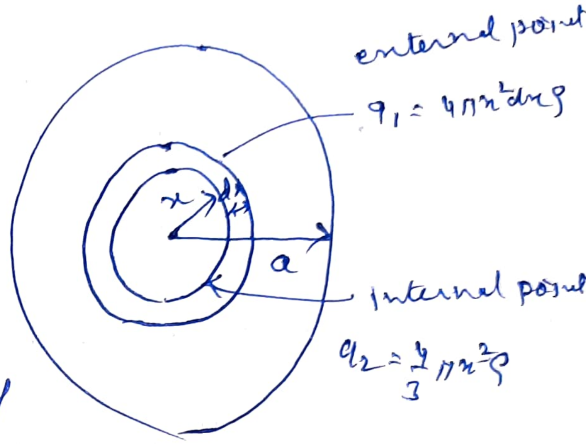
{ $\frac{1}{2}$ is used since in the summation each term appear twice.

Potential energy of a Spherical charge

Let sphere of radius a carries charge Q

charge density

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3} \quad -(1)$$



Let spherical charge may be formed of a large no. of thin cocentric shells

Consider one such shell of radius x and thickness dx

the Volume of shell = $4\pi x^2 dx$

charge on shell = $q_1 = 4\pi x^2 dx \rho$

charge on the sphere of a radius x is $\frac{4}{3}\pi x^3 \rho$

Therefore potential energy of sphere of radius x and spherical shell of radius x and thickness dx is

$$dU = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{x}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{(4\pi x^2 dx \rho)(\frac{4}{3}\pi x^3 \rho)}{x}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{16}{3}\cdot\pi^2\rho^2 x^4 dx$$

total potential energy of spherical charge

$$U = \int_0^a \frac{1}{4\pi\epsilon_0}\frac{16}{3}\pi^2\rho^2 x^4 dx$$

$$= \frac{1}{4\pi\epsilon_0}\frac{16}{3}\pi^2\rho^2\left[\frac{x^5}{5}\right]_0^a = \frac{1}{4\pi\epsilon_0}\cdot\frac{16}{3}\pi^2\rho^2\frac{a^5}{5}$$

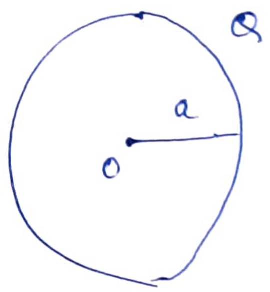
$$U = \frac{1}{4\pi\epsilon_0} \cdot \frac{16}{3}\pi^2 \left(\frac{Q}{\frac{4}{3}\pi a^3}\right)^2 \frac{a^5}{5}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{3}{5} \frac{Q^2}{a}$$

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{Q^2}{a}\right) = \frac{3}{20\pi\epsilon_0}\frac{Q^2}{a}$$

Potential energy of a charged conducting sphere

Let conducting sphere of radius a having charge Q. The charge is on the surface of conductor.



Let the charge is given in small steps when the potential on the conductor charge q on conductor

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{a}$$

Now the charge increases from q to q+dq

the work done

$$dW = \phi \cdot dq = \frac{1}{4\pi\epsilon_0}\frac{q}{a}dq$$

Total work done when charge given from 0 to Q

is

$$W = \int dW = \int_0^Q \frac{1}{4\pi\epsilon_0}\frac{q}{a}dq$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{a}\left[\frac{q^2}{2}\right]_0^Q$$

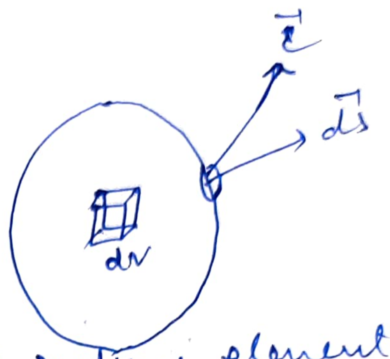
$$= \frac{1}{4\pi\epsilon_0}\frac{Q^2}{2a}$$

the potential energy is

$$U = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{2a}$$

Differential form of Gauss theorem

from Gauss theorem



$$\oint_S \vec{E}\cdot\vec{ds} = \frac{1}{\epsilon_0} \times \text{Net charge enclosed}$$

$$= \frac{1}{\epsilon_0} \Sigma q \quad —(1)$$

If ρ is volume density at small volume element dv

and V is total volume enclosed by surface S

then net charge enclosed by surface S is

$$\Sigma q = \int_V \rho \, dv$$

from eqn (1)

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{1}{\epsilon_0} \int_V \rho \, dv \quad —(2)$$

using Gauss's divergence theorem

$$\oint_S \vec{E}\cdot\vec{ds} = \int_V (\nabla\cdot\vec{E})\, dv$$

from eqn (2)

$$\oint_S E\cdot ds = \int_V (\nabla\cdot\vec{E})\, dv = \frac{1}{\epsilon_0}\int_V \rho \, dv$$

$$\int_V (\nabla\cdot\vec{E} - \rho/\epsilon_0)\, dv = 0$$

$$dv \neq 0$$

$$\text{div}\,\vec{E} - \rho/\epsilon_0 = 0$$

$$\text{div}\, E = \rho/\epsilon_0$$

$$\nabla\cdot E = \rho/\epsilon_0$$

This is Gauss's theorem in differential form

Laplace Poission Equations

If $\vec{E}$ and ϕ are the electric field strength and electric potential at any point respect.,

then
$$E = -\nabla\phi = -\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) \quad \text{—(1)}$$

(i) Poisson eq^n

differential form of Gauss theorem

$$\operatorname{div}\vec{E} = \rho/\epsilon_0$$

using ~~from~~ eq^n (1)

$$\operatorname{div}(-\operatorname{grad}\phi) = \rho/\epsilon_0$$

$$\operatorname{div}\operatorname{grad}\phi = \nabla\cdot(\nabla\phi) = -\rho/\epsilon_0$$

$$\nabla\cdot\nabla\phi = -\rho/\epsilon_0$$

But $\nabla\cdot\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \nabla^2$$

is called laplacian operator

$$\nabla^2\phi = -\rho/\epsilon_0$$

This eq^n is known as Poisson eq^n

(ii) Laplace eq^n

If there is no free charge in the region i.e. charge free region, then $\rho = 0$ so that Poisson eq^n becomes

$$\nabla^2\phi = 0$$

This is Laplace eq^n

Laplace eqⁿ in spherical polar co-ordinate (r, θ, ϕ)

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2} = 0$$

in cylindrical coordinate (r, θ, z)

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Poisson eqⁿ in spherical coordinate (r, θ, ϕ)

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2} = -\rho/\epsilon_0$$

and in spherical coordinate (r, θ, z)

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = -\rho/\epsilon_0$$

Two properties for the solution of laplace eqⁿ

(i) Super-Position theorem

If the function $\phi_1, \phi_2, \phi_3 \ldots \phi_n$, are the solution of laplace's equation, then the the linear superposition of these functions will also be a solution of laplace eqⁿ

The linear superposition of given function

$$\phi = \sum_{i=1}^{n} c_i \phi_i = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \ldots c_n\phi_n$$

where c_i's are constant

Proof Since $\phi_1, \phi_2, \phi_3 \dots \phi_3$ are the solution of laplace's eq^n

$$\nabla^2\phi = 0$$

then

$$\nabla^2\phi_1 = \nabla^2\phi_2 = \dots = \nabla^2\phi_n = 0$$

$$\nabla^2\phi = \nabla^2(c_1\phi_1 + c_2\phi_2 + \dots c_n\phi_n)$$

$$= c_1\nabla^2\phi_1 + c_2\nabla^2\phi_2 + \dots c_n\nabla^2\phi_n$$

$$\nabla^2\phi = 0$$

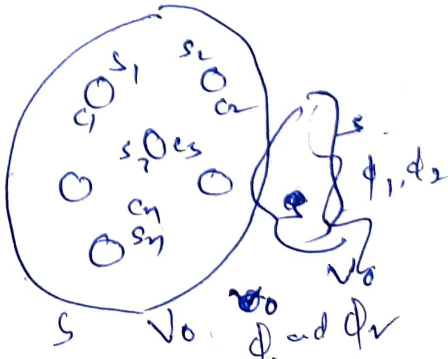
i.e. $\phi = \sum_{i=1}^{n} c_i\phi_i$ is also the solution of laplace eq^n

(ii) <u>Uniqueness theorem</u>

It states that Laplace's eq^n $\nabla^2\phi = 0$ satisfying given boundary conditions has one and only one solution (i.e. unique solution)

proof

Consider a closed volume V_0 exterior to surfaces $S_1, S_2 \dots S_n$ of the various conductors $C_1, C_2 \dots C_n$ and bounded on outside by surfaces



If ϕ_1 and ϕ_2 are two different solutions of laplace eq^n in V_0, which satisfy the same boundary conditions on surfaces $S, S_1, S_2 \dots S_n$.

These bondary conditions may be specified by assigning values of either $\phi_1 = \phi_2$ or $(\nabla\phi_1)_n = (\nabla\phi_2)_n$ (normal component of gradient i.e. $\partial\phi/\partial n$)

If $\phi\nabla\phi$ is a vector function

then Gauss's divergence theorem

$$\int_{S+S_1+S_2\cdots S_n} \phi\nabla\phi \cdot ds = \int_{V_0} \nabla\cdot(\nabla\phi\nabla)\, dV \quad (1)$$

From vector identity

$$\nabla\cdot(\phi\nabla\phi) = \phi\nabla^2\phi + (\nabla\phi)^2$$

from eqn (2)

$$\int_{S+S_1+S_2+\cdots S_n} \phi\nabla\phi \cdot ds = \int_{V_0} \{\phi\nabla^2\phi + (\nabla\phi)^2\}\, dV \quad (2)$$

But laplace eqn $\nabla^2\phi = 0$ at all point in V_0

from eqn (2)

$$\int_{S+S_1+S_2-\cdots+S_n} \phi\nabla\phi \cdot ds = \int_{V_0} (\nabla\phi)^2\, dV \quad (3)$$

As ϕ_1 and ϕ_2 are two solutions of laplace eqn in V_0

hence from superposition theorem their linear combination

$\varphi = \phi_1 - \phi_2$ is also a solution of laplace eqn

satisfying same boundary conditions

from eqn (3)

$$\int (\phi_1-\phi_2)\nabla(\phi_1-\phi_2)\cdot ds = \int (\nabla(\phi_1-\phi_2))^2\, dV \quad (4)$$

Either boundary condition equality of potential $\phi_1 = \phi_2$

or $(\nabla\phi_1)_n = (\nabla\phi_2)_n$ assures that L.H.S. of eqn (4) vanishes

$$\int_{v_0} (\nabla(\phi_1-\phi_2))^2 dv = 0 \quad \text{—(5)}$$

Since $dv \neq 0$

therefore $\nabla(\phi_1-\phi_2) = 0$

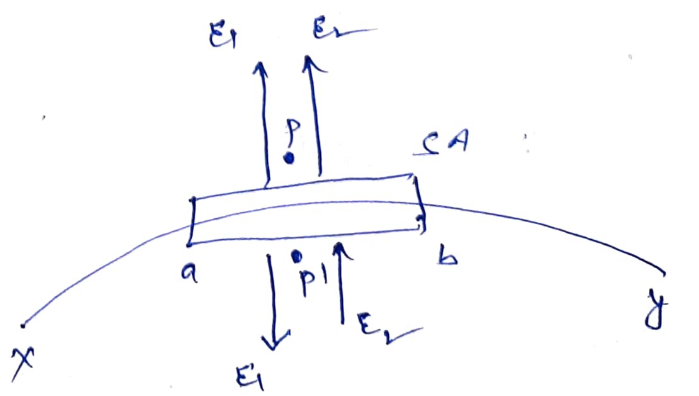
$$\nabla\phi_1 = \nabla\phi_2 \quad \text{—(6)}$$

then $\phi_1 = \phi_2 + \text{Constant}$ (7)

Thus two potential that were assumed different, yet satisfying the same boundary conditions can ~~other~~ differ at most by an additive constant, which make no contribution to the gradient, therefore these potentials will give the same electric field distribution.

Under the given boundary conditions Laplace's eqn has unique solution.

Force on a surface charge and energy Associated with an electric field



In a charge conductor, the total charge resides on its surface, since similar charges ~~on any~~ repel each other, the charge on any part of surface is repelled by the charge on remaining surface of conductor. This mutual repulsion b/w like charges on the surface of charged conductor causes a tension on the surface of conductor. Thus the charged conductor experiences a mechanical force whose magnitude per unit area, we have to determine.

Let XY be the surface of a charged conductor of charged density σ placed in free space.

Consider a small element ab of area δA on the surface of conductor and point P just outside it.

~~A~~ Electric field strength at P is normal to the surface is

$$E = \sigma/\epsilon_0 \quad \text{—(1)}$$

It is assumed that this intensity is partly due to the charge on element ab and partly due to the charge on remaining surface.

If E_1 and E_2 are these field intensity at P respectively

$$E = E_1 + E_2 = \sigma/\epsilon_0 \quad \text{—(2)}$$

Now consider a point P' just near the element ab inside the conductor. Field at P' may also be assumed to be partly due to charge on element ab and partly due remaining charge.

the field at p' due to element ab has same magnitude (E_1) as that point p, but opposite in direction. The E_2 due to the charge on the remaining surface surface has the same magnitude and direction as that at p, since p and p' are very close to each other. Thus resultant intensity at p' is $E_2 - E_1$

But the resultant intensity inside the conductor is zero

$$E_2 - E_1 = 0$$

$$E_1 = E_2$$

from eqⁿ

$$E_1 = E_2 = \frac{\sigma}{2\epsilon_0}$$

Thus the field strength on element ab due to the charge on remaining surface is $E_2 = \sigma/2\epsilon_0$.

This is the mechanical force per unit charge on the element ab.

The charge on small element is $\sigma\,\delta A$

the force on element ab of area δA is

$$\frac{\sigma}{2\epsilon_0}\,\sigma.\delta A = \frac{\sigma^2}{2\epsilon_0}\,\delta A$$

Thus mechanical force per unit area $= \frac{\sigma^2}{2\epsilon_0}$

This force acts out normal to the surface

As $E = \sigma/\epsilon_0 \Rightarrow \sigma = \epsilon_0 E$

$$\frac{F}{\delta A} = \frac{(\epsilon_0 E)^2}{2\epsilon_0} = \frac{1}{2}\epsilon_0 E^2$$

Therefore mechanical force per unit area on the surface of charged conductor acting to the normal

$$= \frac{1}{2}\epsilon_0 E^2$$

Energy associated with an electric field

The outward mechanical force on a small charge element of area δA is

$$F = \frac{\epsilon_0 E^2}{2}\delta A$$

If this small element is displaced normally through a distance δy in the opposite direction of force the work done is

$$\delta W = \left(\frac{\epsilon_0 E^2}{2}\delta A\right)\delta y$$

The energy thus extended can be regarded as stored in additional volume (increase in volume) of electric field is

$$\delta v = \delta A\,\delta y$$

This work done increasing δA volume of electric field

$$\delta W = \frac{\epsilon_0 E^2}{2}(\delta A\,\delta y) = \frac{\epsilon_0 E^2}{2}\cdot\delta v$$

Hence work done per unit volume of field is given as

$$\frac{dW}{dV} = \epsilon_0 E^2/2 \quad J/m^3$$ $\}$ energy of storing in field.

Method of Electrical Images

Lord Kelvin invented the method of electrical images for solving many special problems ~~of~~ in electrostatics with great simplicity

The method of electrical images is to locate either a point charge or a set of point charges or a simplified charge distribution known as electrical image in place of complicated charge distribution such that by their electrical effects the boundary conditions of the ~~p~~ problem remain the same as with the charge distribution

The general principle underlying the method of electrical images is given below

(i) In a system of n point charges $q_1, q_2 \cdots q_n$ the potential at any point is given by

$$\phi = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} \quad \text{— (1)}$$

(ii) The surface of zero potential (grounded conductor) is the ~~fo~~ locus of points for which

$$\phi = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i} = 0 \quad \text{— (2)}$$

(iii) In any problem involving the system of charges $q_1, q_2 \cdots q_i$ and the grounded conductor may be replaced by system of charges $q_{j+1}, q_{j+2} \cdots q_n$ constituting the image if the image charge and the system of real charges combine to produce zero potential over the surface occupied by the conductor.

eqn (2) can be written as

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} + \frac{1}{4\pi\epsilon_0}\sum_{i=j+1}^{n}\frac{q_i}{r_i} = 0 \quad -(3)$$

This eqn represents the condition of potential function over the surface occupied by grounded conductor replaced by electrical image constituted by system of charges $q_{j+1}, q_{j+2} \cdots q_n$.

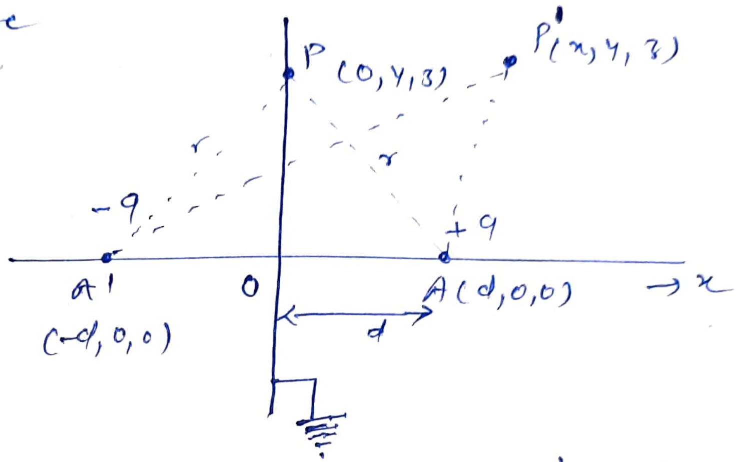
(iv) If the conductor ~~[illegible]~~ is at constant potential $\phi \neq 0$, the system of charges and its image must combine to produce the equal potential ϕ.

Eqn (3) can be written as

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} + \frac{1}{4\pi\epsilon_0}\sum_{i=j+1}^{n}\frac{q_i}{r_i} \quad -(4)$$

This eqn represents the condition of potential function over the surface occupied by conductor replaced by image constituted by system of charges $q_{j+1}, q_{j+2} \cdots q_n$

A point charge near an infinite conducting plane



Let a point charge +q, be placed on x-axis at a distance d to the right of infinite conducting plane placed at $x=0$

The electric potential at every point of conducting plane is zero due grounding plane

Thus the boundary conditions are

(i) The ϕ at every point of conductor is zero
$$\phi = 0 \text{ at } x = 0$$

(ii) The ϕ at infinitely far points is zero
$$\phi = 0, \text{ at } r = \infty$$

These conditions can be satisfied if we replace grounded conductor plane by a point charge (image charge) $-q$ is placed at a distance d to the left side of plane such that line joining the image charge to the right charge intersects the plane of the conductor at right angle.

(i) A point P(0, y, z) on the ~~surface~~ plane conductor, ϕ due to real charge +q and image charge $-q$ is

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{PA} - \frac{q}{PA'}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} - \frac{q}{r}\right] = 0$$

which is first boundary condition of problem

(ii) The charge $-q$ placed at point A' does not affect the condition of electric potential at infinitely far points.

The replacement of grounded conducting plane by the charge $-q$ at A' fulfills the given boundary condition and therefore, according to uniqueness theorem, is the only solution of the problem.

The image charge $-q$ at A' and real charge q at A will produce same field on right hand side of plane as produced due to real charge $+q$ and charge induced on the grounded conductor.

With the help of image charge $(-q)$, we can calculate the surface density of induced charge ~~density~~ on the conductor, force b/w real charge and grounded conducting plane, electric field, potential at any point due to real charge and grounded conducting plane.

Force b/w point charge +q and grounded conductor plane

The force b/w them is same as that b/w charge +q at A and image charge −q at A'. Thus

$$F = \frac{1}{4\pi\epsilon_0} \frac{q(-q)}{(AA')^2} = \frac{1}{4\pi\epsilon_0} \frac{(-q^2)}{(2d)^2}$$

$$= \frac{-1}{4\pi\epsilon_0} \frac{q^2}{4d^2}\ N \quad -(1)$$

Force b/w them is attractive in nature.

Electric ~~field~~ potential at point P' due to point charge q and grounded plane conductor

The potential at P'(x, y, z) due to them is same as that due to charge +q at A and image charge −q at A'. Thus

$$\phi(P') = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{AP'} - \frac{q}{A'P'}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right] \quad -(2)$$

If $x = 0$, $\phi(P) = 0$

If $x = \infty$ from eqn (2) $\phi = 0$

Thus the boundary condition $\phi = 0$, at $x = \infty$ is also satisfied if the grounded plane conductor is replaced by point charge −q at A'

The electric field component at $P(x, y, z)$

$$E_x = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right\}\right]$$

$$E_x = \frac{q}{4\pi\epsilon_0}\left[\frac{(x-d)}{\{(x-d)^2+y^2+z^2\}^{3/2}} - \frac{x+d}{\{(x+d)^2+y^2+z^2\}^{3/2}}\right] \quad -(3)$$

$$E_y = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}\left[\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right\}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{y}{\{(x-d)^2+y^2+z^2\}^{3/2}} - \frac{y}{\{(x+d)^2+y^2+z^2\}^{3/2}}\right] \quad -(4)$$

Similarly

$$E_z = \frac{q}{4\pi\epsilon_0}\left[\frac{z}{\{(x-d)^2+y^2+z^2\}^{3/2}} - \frac{z}{\{(x+d)^2+y^2+z^2\}^{3/2}}\right] \quad -(5)$$

Then $\vec{E} = \hat{i}E_x + \hat{j}E_y + \hat{k}E_z$ $\quad -(6)$

Electric field at conductor

field at point $P(0, y, z)$ is

$$(E_x)_{x=0} = -\left(\frac{\partial \phi}{\partial x}\right)_{x=0} = \frac{q}{4\pi\epsilon_0}\left[\frac{-d}{(d^2+y^2+z^2)^{3/2}} - \frac{d}{(d^2+y^2+z^2)^{3/2}}\right]$$

$$(E_x)_{x=0} = -\frac{qd}{2\pi\epsilon_0 (d^2+y^2+z^2)^{3/2}} \quad -(7)$$

$$(E_y)_{x=0} = -\left(\frac{\partial \phi}{\partial y}\right)_{y=0} = \frac{q}{4\pi\epsilon_0}\left[\frac{y}{(d^2+y^2+z^2)^{3/2}} - \frac{y}{(d^2+y^2+z^2)^{3/2}}\right]$$

$$(E_y)_{x=0} = 0 \quad -(8)$$

Similar $(E_z)_{x=0} = 0$

The electric field at point P(0, y, z) an the conductor has magnitude $\frac{qd}{4\pi\epsilon_0 (d^2+y^2+z^2)^{3/2}}$ and is directed normally towards the conductor. Therefore

$$E(P) = -\frac{qd}{2\pi\epsilon_0 (d^2+y^2+z^2)^{3/2}}\,\hat{i} \quad \text{— (9)}$$

$\hat{i}$ is unit vector along x-axis.

Surface charge density on the conductor

If σ is surface ~~charge~~ charge density of induced charge on the conducting plane at P(x, y, z), the electric field at P can be given as

$$E = (E_x)_{x=0} = \frac{\sigma}{\epsilon_0} \quad \text{— (10)}$$

Then

$$\sigma = \epsilon_0 (E_x)_{x=0}$$

$$= \epsilon_0 \left[\frac{-qd}{2\pi\epsilon_0 (d^2+y^2+z^2)^{3/2}} \right]$$

$$= \frac{-qd}{2\pi (d^2+y^2+z^2)^{3/2}}$$

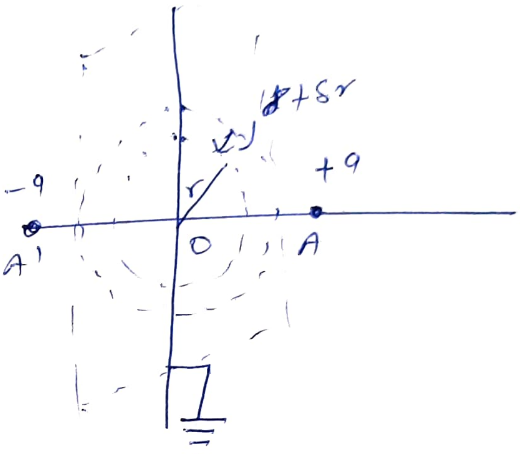
$$= -\frac{qd}{2\pi (d^2+r^2)}$$

where $r^2 = y^2+z^2$

Total charge induced on the conducting plane

Consider a ring area on the plane ~~across~~ conductor around O with radii r and r+δr.

The charge δQi on this elementary area



$$\delta Q_i = 2\pi r \delta r\, \sigma$$

$$= 2\pi r\, \delta r \left[\frac{-qd}{2\pi (r^2+d^2)^{3/2}}\right]$$

$$= \frac{-q d\, r.\delta r}{(r^2+d^2)^{3/2}}$$

Total charge induced on plane

$$Q_i = -\int_0^\infty \frac{qd\, r\, \delta r}{(r^2+d^2)^{3/2}} = -qd \int_0^\infty \frac{r\, \delta r}{(r^2+d^2)^{3/2}}$$

put $r = d\tan\theta$

$\delta r = d\sec^2\theta\, d\theta$

Then $$Q_i = -qd \int_0^{\pi/2} \frac{d\tan\theta\, d\sec^2\theta\, \delta\theta}{(d^2\tan^2\theta + d^2)^{3/2}}$$

$$= -qd \int_0^{\pi/2} \frac{d\tan\theta . d\sec^2\theta\, \delta\theta}{d^2\sec^3\theta}$$

$$= -q \int_0^{\pi/2} \sin\theta\, \delta\theta$$

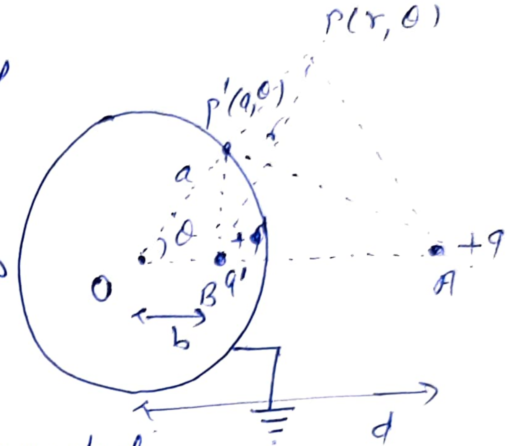
$$= -q\left[-\cos\theta\right]_0^{\pi/2}$$

$$= -q$$

Thus charge induced on earthed plane conductor is $-q$

A point charge near the conducting sphere

Consider a point charge +q placed at a distance d from centre O of grounded conducting sphere of radius a ~~so~~ (a < d)



Since the conducting sphere is grounded, the boundary conditions are

(i) potential ϕ at every point on surface of sphere is zero

$\phi = 0$ where $r = a$

(ii) potential at infinity is zero

$\phi = 0$ where $r = \infty$

Let the image charge q' placed at point B (OB = b) when combined with charge +q leads to zero potential at any point P' on the surface of conductor

The potential at P' due to charge +q and q' is

$$\phi(P') = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{AP'} + \frac{q'}{BP'}\right] = 0$$

then $\frac{q}{AP'} + \frac{q'}{BP'} = 0$

$$q' = -q\frac{BP'}{AP'} \quad \text{— (1)}$$

from Δ OAP'

$$(AP')^2 = (OP')^2 + (OA)^2 - 2OP'(OA)\cos\theta$$

$$(AP') = \sqrt{(a^2+d^2-2ad\cos\theta)}$$

Similarly from Δ OBP'

$$(BP')^2 = (OP')^2 + (OB)^2 - 2(OP')\, OB \cos\theta$$

$$BP' = \sqrt{a^2+b^2-2ab\cos\theta}$$

from eqn (1)

$$q' = -q\,\frac{\sqrt{a^2+b^2-2ab\cos\theta}}{\sqrt{a^2+d^2-2ad\cos\theta}} \quad \text{--- (2)}$$

Let us choose $b = \frac{a^2}{d}$

Then $$q' = -q\,\frac{\sqrt{a^2+(a^2/d)^2-2a\cdot\frac{a^2}{d}\cos\theta}}{\sqrt{a^2+d^2-2ad\cos\theta}}$$

$$= -q\,\frac{a}{d}\left[\left\{\frac{d^2+a^2-2ad\cos\theta}{a^2+d^2-2ad\cos\theta}\right\}^{1/2}\right]$$

$$q' = -q(a/d) \quad \text{--- (3)}$$

Thus the image charge is a point charge of magnitude $-q(a/d)$ placed at a distance (a^2/d) from the centre of sphere along the line joining the centre sphere to real point charge $+q$

The potential at $P(r,\theta)$ due to $+q$ and q' (due to point charge $+q$ place near on earthed conducting sphere of radius a is

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{AP} + \frac{q'}{BP}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(r^2+d^2-2rd\cos\theta)^{1/2}} + \frac{q'}{(r^2+b^2-2rb\cos\theta)^{1/2}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{q(a/d)}{\sqrt{r^2+(a^2/d)^2-2r.\frac{a^2}{d}\cos\theta}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{q}{\sqrt{\frac{r^2d^2}{a^2}+a^2-2rd\cos\theta}}\right] \quad \text{--- (4)}$$

Electric field component at $p(r,\theta)$ is

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\frac{r-d\cos\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{(rd^2/a^2 - d\cos\theta)}{\left(\frac{r^2d^2}{a^2}+a^2-2rd\cos\theta\right)^{3/2}}\right] \quad \text{--- (5)}$$

and

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{q}{4\pi\epsilon_0}\left[\frac{d\sin\theta}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{d\sin\theta}{\left(\frac{r^2d^2}{a^2}+a^2-2rd\cos\theta\right)^{3/2}}\right] \quad \text{--- (6)}$$

Electric field on the surface of sphere

field at $P'(a,\theta)$ is

$$(E_r)_{r=a} = -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \frac{q}{4\pi\epsilon_0}\left[\frac{a-d^2/a}{(d^2+a^2-2ad\cos\theta)^{3/2}}\right]$$

$$(E_r)_{r=a} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a}\cdot\frac{a^2-d^2}{(a^2+d^2-2ad\cos\theta)}\right] \quad \text{--- (7)}$$

$$(E_\theta)_{r=a} = 0 \quad \text{--- (8)}$$

The field at surface is

$$E = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{a} \frac{a^2-d^2}{(a^2+d^2-2ad\cos\theta)^{3/2}}$$

is along the line joining from centre of sphere to the point p' on surface.

Surface ~~charge~~ density of induced charge on sphere.

If σ is surface density of induced charge on p' in

$$E = (E_r)_{r=a} = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \epsilon_0 (E_r)_{r=a}$$

$$= -\frac{q}{4\pi a} \frac{d^2-a^2}{(a^2+d^2-2ad\cos\theta)^{3/2}} \quad \text{—— (9)}$$

The force b/w sphere and point charge.

The force b/w sphere of radius a and point charge +q is same as that b/w point charge +q and $q' = (-q\,a/d)$

Thus

$$F = \frac{1}{4\pi\epsilon_0} \frac{qq'}{(AB)^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q(-q\,a/d)}{(d-b)^2} = \frac{q(-q\,a/d)}{(d-a^2/d)^2}$$

$$= -\frac{1}{4\pi\epsilon_0} \cdot \frac{q^2(ad)}{(d^2-a^2)^2} \text{ Newton.}$$

-ive sign indicates that force is attractive