Th Show that for each $a \in G$ and $x \in G$, the order of $x^{-1}ax$ = order of a where G is a group.

Proof :- Let us take $a, x \in G$, then consider

$$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax)$$
$$= x^{-1}axx^{-1}ax$$
$$= x^{-1}a(xx^{-1})ax$$
$$= x^{-1}aeax \qquad \because xx^{-1} = e$$
$$= x^{-1}a^2x \qquad \text{——— (i)}$$

Similarly $(x^{-1}ax)^3 = x^{-1}a^3x$ ——— (ii)

Further if we assume that

$$(x^{-1}ax)^{n-1} = x^{-1}a^{n-1}x$$

then $(x^{-1}ax)^n = (x^{-1}ax)^{n-1}(x^{-1}ax)$ where n is an integer

$$= x^{-1}a^{n-1}x \cdot x^{-1}ax$$
$$= x^{-1}a^{n-1}eax$$
$$= x^{-1}a^nx \qquad \text{——— (iii)}$$

$\Rightarrow \boxed{(x^{-1}ax)^n = x^{-1}a^nx} \quad \forall\, n \in N$

Let us assume that order of a is n

$\Rightarrow a^n = e$

Then (iii) $\Rightarrow (x^{-1}ax)^n = x^{-1}ex = e$

$\Rightarrow (x^{-1}ax)$ is also of finite order

and $o(x^{-1}ax)^n \leq n$ ——— (iv)

Again let $o(x^{-1}ax) = m$

$\Rightarrow (x^{-1}ax)^m = e$

$\Rightarrow x^{-1}a^m x = e$

$\Rightarrow x(x^{-1}a^m x)x^{-1} = xex^{-1}$

$\Rightarrow a^m = e$

$\Rightarrow o(a) \leq m$

$\Rightarrow \boxed{n \leq m}$ —— (V)

from (iv) & (V) $m = n$

ie. $o(x^{-1}ax) = o(a)$ <u>H.P.</u>

<u>Th</u> Let G be a group and $a^2 = e$ $\forall a \in G$, then prove that G is abelian.

<u>Proof</u>: Let us take two elements $a, b \in G$.

then if $a \in G \Rightarrow a^2 = e$

$\Rightarrow aa = e$

$\Rightarrow a^{-1}(aa) = a^{-1}e$

$\Rightarrow (a^{-1}a)a = a^{-1}$

$\Rightarrow ea = a^{-1}$

$\Rightarrow \boxed{a = a^{-1}}$ —— (i)

Again Similarly if $b \in G$

$\Rightarrow \boxed{b = b^{-1}}$ —— (ii)

Now if $a \in G, b \in G \Rightarrow ab \in G$ $\because G$ is a group.

$\Rightarrow (ab)^2 = e$ As per given condition

$\Rightarrow (ab) = (ab)^{-1}$

$\Rightarrow ab = b^{-1}a^{-1}$

$\Rightarrow ab = ba$ using (i) & (ii)

$\Rightarrow$ G is abelian.

Th If $a, b \in G$, then show that $(ab)^2 = a^2b^2$ iff G is abelian.

Prf:— we are to prove that
(i) If G is abelian $\Rightarrow (ab)^2 = a^2b^2$
(ii) If $(ab)^2 = a^2b^2 \Rightarrow$ G is abelian

case (i) let G is abelian

$\Rightarrow ab = ba \quad \forall a, b \in G$

$\Rightarrow (ab)(ab) = (ba)(ab)$

$\Rightarrow (ab)^2 = ba^2b$

$= (a^2b)b \quad \because$ G is abelian

$\Rightarrow \boxed{(ab)^2 = a^2b^2}$ —— (i) H.P.

Again case (ii) let us take $(ab)^2 = a^2b^2$

$\Rightarrow (ab)(ab) = aa\,bb$

$\Rightarrow a(ba)b = a(ab)b$

$\Rightarrow a^{-1}a(ba)bb^{-1} = a^{-1}a(ab)bb^{-1}$

$\Rightarrow e(ba)e = e(ab)e$

$\Rightarrow \boxed{ba = ab}$ —— (ii) H.P.

<u>Th</u> Let for a group G, $ba = a^m b^n$, then prove that $a^m b^{n-2}$, $a^{m-2} b^n$, ab^{-1} are of the same order.

<u>Proof</u>:- Let $a, b \in G$, then, we know that $a^m b^{n-2}$, $a^{m-2} b^n$ and ab^{-1} also belongs to G.

Take $a^m b^{n-2} = a^m b^n b^{-2}$

$= (a^m b^n) b^{-2}$

$= (ba) b^{-2}$

$= b(ab^{-1}) b^{-1}$

$= (b^{-1})^{-1} (ab^{-1}) (b^{-1})$

$\Rightarrow o(a^m b^{n-2}) = o(ab^{-1})$ —— (i) $\quad \because o(x^{-1} a x) = o(a)$

Again take $a^{m-2} b^n = a^{-2} a^m b^n$

$= a^{-1} a^{-1} (a^m b^n)$

$= a^{-1} a^{-1} (ba)$

$= a^{-1} (a^{-1} b) a$

$\Rightarrow o(a^{m-2} b^n) = o(a^{-1} b)$ $\quad \because o(x^{-1} a x) = o(a)$

$= o((a^{-1} b)^{-1})$ $\quad \because o(x) = o(x^{-1})$

$= o(b^{-1} a)$

$= o(e b^{-1} a)$

$= o(a^{-1} a b^{-1} a)$

$= o(a^{-1} (ab^{-1}) a)$

$= o(ab^{-1})$ $\quad \because o(x^{-1} a x) = o(a)$ —— (ii)

From (i) & (ii) $o(a^{m} b^{n-2}) = o(a^{m-2} b^{n}) = o(ab^{-1})$

H.P.

Th. If a group G has four elements, then show that it must be abelian.

Solⁿ. we know that a set of four elements which is group, can taken as Klien's Group-4.

i.e. $K = \{e, a, b, c\}$ for * composition defined by the table as follows

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Here this group is an abelian group:

$\because$ $ab = ba = c$; $ac = ca = b$, $bc = cb = a$

H.P.