pmf : prob. mass fn.

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Unit- II

M_4

Discrete Distribution

Binomial Distribution

Let a trial repeated n-times, we called occurrence of event a success and non-occurrence a failure.
Let p be probability of success and q be probability of failure so that

$$\boxed{p+q=1}$$

* We assume that trials to be independent and p and q are same in every trial.

The probability of x success and (n-x) failure in n-independent trials, in specified order

SSFSFFFS.... FSF

$$P(SSFSFFFS....FSF) = P(S)P(S)P(F)P(S)P(F)P(F)P(F) ---- P(F)P(S)P(F)$$

$$= ppqpqqqp --- qpq$$

$$= \underbrace{(pp---p)}_{(x)\text{times}}\underbrace{(qq---q)}_{(n-x)\text{ times}}$$

$$= p^x q^{n-x}.$$

Prob. $p(x)$ of x success in series of x ind. trial.

$$\{P(X=x)=p_x\} = {}^nC_x p^x q^{n-x} ; x = 0, ...$$

pmf of binomial distribut...

permissible occour.

$$\sum p(x) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$$

$$= q^n + npq^{n-1} + \dots + {}^nC_n p^n$$

$$= (p+q)^n.$$

$$= 1^n$$

$$= 1.$$

* $X \sim B(n, p)$ where n = trial
p = success
are parameters

Physical Distribution of Binomial dist^n

1) n = finite (trials are finite)
2) trial are ind. to each other.
3) Prob. of success p is constant (same) for each trial.

* [2 mutual cases — S and F]

4) Each trial result in 2 ~~outcomes~~ exhaustive mutually disjoint outcomes termed as success and failure

Mean of $X = np$
[Derivation from Book]

Var. of $X = npq$.

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Applications of Binomial Dist^n

(1) Problems relating to tossing of coin or throwing of dice or drawing cards from pack of cards (only with replacement)

$\because$ in without replacement it is dependent so Binomial cond^n not follows.

Q (2) Mean and Variance of Binomial Variate X are 8 and 4. find

i) $P(x=1)$
ii) $P(x=2)$

Sol^n $np = 8$. —(i)
$npq = 4$. —(ii)

$q = \frac{1}{2}$ so $p = \frac{1}{2}$.

from (i) $\boxed{n=16}$.

i) $P(x=1) = {}^{16}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{15} = \frac{16}{2^{16}} = \frac{1}{2^{12}}$

ii) $P(x=2) = {}^{16}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{14} = \frac{16 \times 15}{2 \cdot 2^{16}} = \frac{15 \cdot 2^4}{2^{17}}$

$= \frac{15}{2^{13}}$

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Q ~~find~~ for binomial variable X, find p if $n = 4$ and $P(X=4) = 6P(X=2)$

Sol: $P(X=4) = 6P(X=2)$

for binomial variate X, pmf where $P(X=x) = {}^nC_x p^x q^{n-x}$.

$${}^4C_4 p^4 q^0 = 6 \cdot {}^4C_2 p^2 q^2.$$

$$p^{\cancel{4}2} = \cancel{6}^{3} \cdot \frac{4\times 3}{\cancel{2}\times 1} \cancel{p^2} q^2$$

$$p^2 = 36 q^2 \Rightarrow p = \pm 6q$$

Taking +ve, $\frac{p}{q} = 6$

$p = 6 - 6p$

$7p = 6$

$\boxed{p = \frac{6}{7}}$

Taking -ve

$p = -6q$

$-5p = -6.$

$p = \frac{6}{5} > 1$ which is not possible.

NR

~~$X = npq = \frac{4 \cdot 6 \cdot 1}{7 \cdot 7}$~~

~~$= \frac{24}{49}.$~~

Q With usual notation find p for binomial variate X, if $n = 6$ and $9P(X=4) = P(X=2)$

Sol: for binomial variate X, pmf

$P(X=x) = {}^nC_x p^x q^{n-x}$

NR ~~$9(px$~~ $9P(X=4) = P(X=2)$

$$9 \cdot \cancel{{}^6C_4} p^{4} q^2 = \cancel{{}^6C_2} p^2 q^4.$$

$$9 \cdot p^2 = q^2.$$

$$q = \pm 3p.$$

Central Moment $\rightarrow \mu_r = E(X - E(X))$
Mean.

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$q = 3p$ $\qquad$ $q = -3p$.
$1-p = 3p$. $\qquad$ $1-p = -3p$.
$p = 1/4$ $\qquad$ $1 = -2p$
$p = -1/2$ Not possible.

Recurrence Relation for moments of Binomial Distribution

i.e. $\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right]$

Sol^n $\mu_r = E[(X - E(X))^r]$

$\Rightarrow \mu_r = \sum (x-np)^r \, {}^nC_x p^x q^{n-x}$.

$\frac{d\mu_r}{dp} = \sum\left[(-nr)(x-np)^{r-1} \, {}^nC_x p^x q^{n-x} + \frac{x}{p}(x-np)^r \, {}^nC_x p^{x-1} q^{n-x}\right] +$

~~$\frac{d\mu_r}{dp} = \sum[nr$~~

$\frac{d\mu_r}{dp} = \sum_{x=0}^{n}(-nr)(x-pn)^{r-1} \, {}^nC_x p^x q^{n-x} + \sum_{x=0}^{n}(x-np)^r \, {}^nC_x \left(xp^{x-1} q^{n-x} - (n-x)p^x q^{n-x-1}\right)$

$\frac{d\mu_r}{dp} = -nr\mu_{r-1} + \sum_{x=0}^{n}(x-np)^r \cdot {}^nC_x p^x q^{n-x} \left(\frac{x}{p} -\right.$

$qx - pn + px$
$(1-p)x - pn + px$
$x - px - pn + px$
$x - pn$

$= -nr\mu_{r-1} + \sum_{x=0}^{n}\frac{(x-np)^{r+1} \, {}^nC_x p^x q^{n-x}}{pq}$.

$\frac{d\mu_r}{dp} = -n\mu_{r-1} + \frac{\mu_{r+1}}{pq}$

~~$\mu_1' = 1$~~. $\mu_0 = 1$

~~$\mu_0 = \mu_r + t$~~ $\mu_1 = 0$.

$$\mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right).$$

n=2

$$\mu_2 = pq\left(n\mu_0 + \frac{d\mu_1}{dp}\right). \quad \boxed{\mu_0 = 1}.$$

$\mu_1 = pqn$.

$\boxed{\mu_2 = npq}$

n=3

$$\mu_3 = pq\left[n \cdot 2\mu_1 + \frac{d\mu_2}{dp}\right].$$

~~$\mu_3 = pq[2n + nq]$~~

~~$= npq(2+q)$~~

~~$= npq($~~

$=$

$$\mu_3 = npq(q-p)$$

$$\mu_4 = npq\left[1 + 3pq(n-2)\right]$$

μ_4

Q $X \sim B(n, p)$

show that $E\left[\left(\frac{X}{n} - p\right)^2\right] = \frac{pq}{n}$.

Sol $E\left(\frac{X}{n}\right) = \frac{1}{n} E(X)$.

$= \frac{1}{n} \cdot np = p$.

$$E\left[\left(\frac{X}{n} - p\right)^2\right] = E\left[\left(\frac{X}{n} - E\left(\frac{X}{n}\right)\right)^2\right] = Var\left(\frac{X}{n}\right) = \frac{1}{n^2}(npq)$$

$$= \frac{pq}{n}.$$

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Q. Ten coins are tossed simultaneously. Find prob of getting at least 7 head.

Solⁿ $n = 10 \quad p = \frac{1}{2} \quad q = \frac{1}{2}$

$p(X \geq 7) = {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)$

$= \frac{1}{2^{10}}\left[120 + 45 + 10 + 1\right]$

$= \frac{1}{2^{10}}\left[176\right] = \frac{176}{1024}$

Q2 Mean and Variance of X be 4 and $\frac{4}{3}$ for $P(X \geq 1)$.

Solⁿ $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

~~$1 - {}^{n}C_0$~~

$= 1 - {}^{n}C_0 p^0 q^{n-0}$

$np = 4$

$npq = \frac{4}{3}$

$q = \frac{1}{3} \quad p = \frac{2}{3} \quad n = 6$

$= 1 - {}^{6}C_0 p^0 q^{6}$

$= 1 - \left(\frac{1}{3}\right)^{6}$

Mgf of Binomial $M_x(t) = (q+pe^t)^n$

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Recurrence Relation for Cumulants of Binomial Dist^n

Prove $K_{r+1} = pq \frac{dk_r}{dp}$.

Proof $K_r = \left[\frac{d^r}{dt^r} K(t)\right]_{t=0}$

$^{10}C_{10}(\frac{1}{2})^{10}$

mgf of Binomial.

$M_x(t) = E(e^{tx})$

$M_x(t) = \sum e^{tx} \, {}^nC_x p^x q^{n-x}$.

$= \sum {}^nC_x (pe^t) q^{n-x}$.

$= (q+pe^t)^n$.

Characteristic f^n

$\phi_x(t) = (q+pe^{it})^n$

Now $K_r = \left[\frac{d^r}{dt^r} \log M(t)\right]_{t=0}$.

$= \left[\frac{d^r}{dt^r} \log(q+pe^t)^n\right]_{t=0}$

$K_r = n\left[\frac{d^r}{dt^r} \log(q+pe^t)\right]_{t=0}$

$\frac{dK_r}{dp} = n\left[\frac{d^r}{dt^r} \frac{(-1+e^t)}{(q+pe^t)}\right]_{t=0}$

$d(x+y) = d(x) \ldots$

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Now, $K_{r+1} = \left[\frac{d^{r+1}}{dt^{r+1}} \log(q+pe^t)\right]_{t=0}$

$$K_{r+1} = n\left[\frac{d^r}{dt^r}\frac{d}{dt}\log(q+pe^t)\right]_{t=0}$$

$$K_{r+1} = n\left[\frac{d^r}{dt^r}\left(\frac{pe^t}{q+pe^t}\right)\right]_{t=0}.$$

$\Rightarrow$ $K_{r+1} = n\left[\frac{d^r}{dt^r}\left(\frac{q+pe^t-q}{q+pe^t}\right)\right]_{t=0}$

$\Rightarrow$ $K_{r+1} = n\left[\frac{d^r}{dt^r}\left[1-\frac{q}{q+pe^t}\right]\right]_{t=0}$

$\Rightarrow$ $K_{r+1} = -nq\left[\frac{d^r}{dt^r}\left(\frac{1}{q+pe^t}\right)\right]_{t=0}$

$K_{r+1} - pq\frac{dk_r}{dp} = -nq\frac{d^r}{dt^r}\left(\frac{1}{q+pe^t}\right)_{t=0}.$

$$-pqn\frac{d^r}{dt^r}\left(\frac{-1+e^t}{q+pe^t}\right)_{t=0}$$

$$= -nq\left[\frac{d^r}{dt^r}\left(\frac{1-p+pe^t}{q+pe^t}\right)\right]_{t=0}$$

$$= -nq\frac{d^r}{dt^r}\left[\frac{q+pe^t}{q+pe^t}\right]_{t=0}$$

$$= -nq\left[\frac{d^r}{dt^r}(1)\right]_{t=0}$$

$$= 0.$$

Mean, variance, Characteristic f^n, mgf,
(Replace with it in mgf)

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$\therefore \boxed{k_{r+1} = pq \frac{dk_r}{dp}}$

Additive Property of Binomial Dist^r
(Reproductive)

Let $X \sim B(n_1, p_1)$ and $Y \sim B(n_2, p_2)$
be two independent Binomial variable

$M_{X+Y}(t) = M_X(t) M_Y(t)$ $\{\because$ X and Y are ind. Binomial Variate$\}$

$= (q_1 + p_1 e^t)^{n_1} (q_2 + p_2 e^t)^{n_2}$

It cannot be expressed as in the form $(q+pe^t)^n$

By uniqueness theorem $X+Y$ is not Binomial Variant

* However, if $p_1 = p_2$ ~~$= q_1 = q_2$~~ $= p$
and $q_1 = 1 - p_1 = 1 - p = q$
$q_2 = 1 - p_2 = 1 - p = q$

$M_{X+Y}(t) = (q + pe^t)^{n_1} (q + pe^t)^{n_2}$
$= (q + pe^t)^{n_1 + n_2}$

$\therefore$ ~~X+Y follow~~ $(X+Y) \sim B(n_1 + n_2, p)$

Q. Given $z = \log_e \left(\frac{p}{q}\right)$ where $X \sim B(n, p)$

Prove

i) $\frac{d}{dt} k(t) = n\left[1 + e^{-(z+t)}\right]^{-1}$

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ii) $k_r = n \frac{d^{r-1}}{dz^{r-1}} P.$

(E) iii) Prove $k_{r+1} = pq \frac{dk_r}{dp}.$

iv) $k_{r+1} = \frac{dk_r}{dz}.$

Sol

$$\frac{d}{dt} K(t) = \frac{d}{dt}\left[\log M_x(t)\right]$$

$$= \frac{d}{dt}\left[\log (q+pe^t)^n\right]$$

$$= n \frac{d}{dt} \log (q+pe^t)$$

$$= n \left(\frac{pe^t}{q+pe^t}\right)$$

$$= n \frac{1}{\left(\frac{q}{p}e^{-t}+1\right)} \quad \text{(Divide by } pe^t)$$

$$= n \frac{1}{e^{-z-t}+1}$$

$$= n\left[1+e^{-(z+t)}\right]^{-1}.$$

ii) $\cancel{K_r = \frac{d^r}{dt^r}\left\{\left[\log (q+pe^t)^n\right]\right\}_{t=0}}$

$$K_r = \left[\frac{d^r}{dt^r} K(t)\right]_{t=0}$$

$$= \left[\frac{d^{r-1}}{dt^{r-1}} \frac{d}{dt} K(t)\right]_{t=0}$$

$$= \frac{d^{r-1}}{dt^{r-1}} \frac{d}{dt}\left\{n\left[1+e^{-(z+t)}\right]^{-1}\right\}_{t=}$$

$$K_r = n \frac{d^{r-1}}{dt^{r-1}} \left[\frac{e^{z+t}}{e^{z+t}+1} \right]_{t=0}.$$

$\frac{e^{z+t}}{e^{z+t}+1}$ is symmetric in z and t.

so $\frac{d^{r-1}}{dt^{r-1}} \left[\frac{e^{z+t}}{e^{z+t}+1} \right]_{t=0} = \frac{d^{r-1}}{dz^{r-1}} \left[\frac{e^{z+t}}{e^{z+t}+1} \right]_{t=0}$

$$K_r = n \frac{d^{r-1}}{dz^{r-1}} \left[\frac{e^{z+t}}{e^{z+t}+1} \right]_{t=0}$$

$$K_r = n \frac{d^{r-1}}{dz^{r-1}} \left(\frac{e^z}{e^z+1} \right)$$

$$K_r = n \frac{d^{r-1}}{dz^{r-1}} \left[\frac{p/q}{p/q+1} \right]$$

$$= n \frac{d^{r-1}}{dz^{r-1}} p.$$

ii) $K_{r+1} = pq \frac{dk_r}{dp}$ ~~$n \frac{d^r}{dt^r} \frac{d}{dt} K(t)$~~

$= pq.$

~~$= n \frac{d}{dt} \left[n \frac{d^{r-1}}{} \right.$~~

iii) $K_r = n \frac{d^{r-1}}{dz^{r-1}} p.$

$$\frac{dk_r}{dp} = n \cdot \frac{d}{dp} \left[\frac{d^{r-1}}{dz^{r-1}} p \right]$$

$$= n \frac{d}{dz} \left[\frac{d^{r-1}}{dz^{r-1}} p \right] \frac{dz}{dp}. = n \frac{d^r p}{dz^r} \frac{dz}{dp}. = \frac{k_{r+1}}{pq}$$

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$$k_{r+1} = pq \frac{dk_r}{dp}$$

iv) $\cancel{k_{r+1}} = \cancel{x \frac{d}{dz}\left[\frac{d^{r-1} p}{dz^{r-1}}\right]}$.

$$\frac{dk_r}{dz} = \frac{dk_r}{dp}\frac{dp}{dz} = \frac{\frac{dk_r}{dp}}{\frac{dz}{dp}} = \frac{dk_r}{dp} \Big/ \frac{1}{pq} = pq\frac{dk_r}{dp}$$

$$\Rightarrow \boxed{\frac{dk_r}{dz} = k_{r+1}}$$

H.P.

Q $X \sim B(n,p)$. What is dist^r of $Y = n - X$:

Sol^n $X \sim B(n,p)$ $M_x(t) = E(e^{tx}) = (q+pe^t)^n$

$$M_Y(t) = \cancel{q+p}\ E(e^{tY}) = E\left(e^{t(n-x)}\right)$$

$$= e^{nt} E(e^{-tx})$$

$$= e^{nt} M_x(-t)$$

$$= e^{nt}(q+pe^{-t})^n$$

$$= \left[e^t(q+pe^{-t})\right]^n$$

$$= (p+qe^t)^n$$

$Y = n - X \sim B(n,q)$.

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Q m things are distributed among 'a' men and 'b' women, show that probability no. of things received by men in odd is

$$\frac{1}{2}\left[\frac{(b+a)^m-(b-a)^m}{(b+a)^m}\right]$$

Solⁿ Probability that a thing is received by man $- p = \frac{a}{a+b}$.

women $- q = 1-p$
$= \frac{b}{a+b}$.

Probability that m things exactly x are received by man

$$p(x) = {}^mC_x\, p^x q^{m-x} \quad , x=0,1,\dots$$

Req. prob $P = p(1)+p(3)+p(5)+\dots$
$= {}^mC_1\, p^1 q^{m-1} + {}^mC_3\, p^3 q^{m-3} + {}^mC_5\, p^5 q^{m-5}+\dots$
$=$ ~~mC~~

$$(q+p)^m = q^m + {}^mC_1\, q^{m-1}p + {}^mC_2\, q^{m-2}p^2+\dots$$

$$(q-p)^m = q^m - {}^mC_1\, q^{m-1}p + {}^mC_2\, q^{m-2}p^2+\dots$$

$$\frac{1}{2}\left[(q+p)^m-(q-p)^m\right] = {}^mC_1\, p^1 q^{m-1} + {}^mC_3\, p^3 q^{m-3}+\dots$$

$$P = \frac{1}{2}\left[(q+p)^m-(q-p)^m\right]$$

$q+p = 1$
$q-p = \frac{b-a}{a+b}$.

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$$p = \frac{1}{2}\left[1 - \frac{(b-a)^m}{(b+a)^m}\right]$$

$$p = \frac{1}{2}\left[\frac{(b+a)^m - (b-a)^m}{(b+a)^m}\right]$$

Q Show r^{th} moment μ_r' about origin of binomial dist^r of degree n is

PM'

$$\mu_r' = \left(p\frac{\partial}{\partial p}\right)^r (q+p)^n$$

$-2\sigma < X - \mu < 2\sigma$

$\mu - 2\sigma < X$.

Q. Mode of Binomial Dist^r

i) If $(n+1)p$ is not integer.

Mode is equal to integral part of $(n+1)p$ (UNIMODAL)

ii) $(n+1)p$ is integer.

Let $(n+1)p = m$

then mode is $m, m-1$ (Bimodal)

Q If mean and variance of binomial ~~varia~~ dist^r 4 and 3. Find Mode.

Sol^n

$\cancel{npq} = \quad np = 4 \qquad X \sim B(n,p)$

$npq = 3$.

$q = \frac{3}{4}$

$p = 4$.

$q = \frac{3}{4}$; $p = \frac{1}{4}$; $\cancel{n = \frac{16}{3}} = 5$. $n = 16$

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$(16+1)\cdot\frac{1}{4} = 4.25 = \text{Mode}.$

Thus Mode $= 4$.

Q M.G.F of rv X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$, show

$$P(\mu-2\sigma < X < \mu+2\sigma) = \sum_{x=1}^{5} {}^9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}.$$

Sol^n mgf of Binomial dist^r

$$M_x(t) = (q+pe^t)^n$$

$$q=\frac{2}{3};\ p=\frac{1}{3};\ n=9.$$

μ (Mean) $= np = 9\cdot\frac{1}{3} = 3$; σ^2 (var) $= npq = 2 \Rightarrow \sigma=\sqrt{2}$

~~$P(|X-\mu| > 2\sigma)$~~

$$P(\mu-2\sigma < x < \mu+2\sigma) = P(3-2\sqrt{2} < X < 3+2\sqrt{2})$$
$$= P(0.2 < X < 5.8)$$
$$= P(1 \le X \le 5).$$
$$= \sum_{x=1}^{5} {}^9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

H.P.

For Binomial Dist^r Prove that Recurrence Relation

$$p(x+1) = \frac{p}{q}\,\frac{n-x}{x+1}\,p(x)$$

$$\frac{p(x+1)}{p(x)} = \frac{{}^nC_{x+1}(p)^{x+1}(q)^{n-x-1}}{{}^nC_x\, p^x q^{n-x}} = \frac{\frac{n!}{(x+1)!\,(n-x-1)!}}{\frac{n!}{x!\,(n-x)!}}\,\frac{p}{q}.$$

$$= \left(\frac{n-x}{x+1}\right) p/q.$$

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$$p(x+1) = \frac{p}{q} \frac{n-x}{x+1} p(x)$$

Poisson Dist^n: A r.v. X is said to follow a poisson dist^n if it assume non-negative value. prob^l mass f^n is

(PMF) $$p(x,\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, \ldots$$

* Poisson Distribution is limiting case of Binomial.

i) No. of trial $n \to \infty$
ii) $p \to 0$.
such that $np = \lambda$ (finite)

* Some application of Poisson

i) No. of death from disease such as heart attack or cancer or due to snake ~~bite~~ bite
ii) No. of suicide reported in particular city
iii) No. of air accidents in some unit of time

For poisson : Mean $= \lambda$
Variance $= \lambda$

~~Poisson D~~

Proof :

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$$B(x,n,p) = {}^nC_x \; p^x (1-p)^{n-x}$$

$$= {}^nC_x \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n} \Big/ \left(1-\frac{\lambda}{n}\right)^x$$

$$= \frac{n(n-1)\ldots n-(x-1)\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}} \left(\frac{p}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)\ldots[n-(x-1)]}{x!} \left(\frac{\lambda/n}{1-\frac{\lambda}{n}}\right)^x \left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\cancel{n^x}\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right]\ldots\left[1-\frac{(x-1)}{n}\right]}{x!} \left[\frac{\lambda^x}{\cancel{n^x}\left(1-\frac{\lambda}{n}\right)^x}\right]\left(1-\frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty} B(x,n,p) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Q If $X \sim P(\lambda)$ s.t

$P(X=2) = 9\,(P=4) + 90\,P(X=6)$

Find mean of X

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \lambda > 0$$

Sol^n
$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{9e^{-\lambda}\lambda^4}{4\times 3\times 2} + \frac{90\, e^{-\lambda}\lambda^6}{6\times 5\times 4\times 3\times 2}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda^4 - \lambda^2 + 4\lambda^2 - 4 = 0$$

$$\lambda^2(\lambda^2-1) + 4(\lambda^2-1) = 0 \qquad \lambda^2 = 1 \qquad \lambda = 1$$

(λ can't be $-$ve)

var $\neq$ -ve.

~~P(x=x)~~

Q If X and Y are independent Poisson variate s.t. $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$
Find varian. $(X-2Y)$

Sol^n $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ where $X \sim P(\lambda)$ $\lambda > 0$

$Y \sim P(\mu)$ $\mu > 0$

• $P(X=1) = P(X=2)$.

$\frac{\cancel{e^{-\lambda}}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$

$\boxed{\lambda = 2}$

• $P(Y=2) = P(Y=3)$.

~~$e^{-\lambda}\lambda^1$~~

$\frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-\mu}\mu^3}{3!}$

$\boxed{\mu = 3}$

$Var(X-2Y) = \cancel{2 - 4.3}$ $Var(X) - 4 Var Y$

$= \cancel{2 - 12}$ $2 + 4 \times 3$

$= \cancel{-10}$ 14.

Q Let $X \sim P(\lambda)$ if $P(X=1) = P(X=2)$
find $P(X=0$ or $1)$

Sol^n $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$P(X=1) = P(X=2)$

$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$ $\Rightarrow \boxed{\lambda = 2}$

Mean and Variance of Binomial → Proof.

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$$P(X=0 \text{ or } 1) = P(X=0) + P(X=1)$$

$$= \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!}$$

$$= \frac{3}{e^2}.$$

Q Obtain Mean and Variance of Poisson distribution.

Sol $X \sim P(\lambda)$

$$p(x) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E(x) = \sum x p(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-\lambda}\lambda}{1!} + \frac{2e^{-\lambda}\lambda^2}{2!} + \frac{3e^{-\lambda}\lambda^3}{3!} \dots$$

$$= e^{-\lambda}\lambda\left[1 + \lambda + \frac{\lambda^2}{2!} + \dots\right]$$

$$= e^{-\lambda}\lambda \cdot e^{\lambda}$$

$$= \lambda.$$

$$V(x) = \sum x^2 p(x) - \left[\sum (x p(x))\right]^2$$

$$\sum x^2 p(x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-\lambda}\lambda}{1!} + \frac{2^2 e^{-\lambda}\lambda^2}{2!} + \frac{3^2 e^{-\lambda}\lambda^3}{3!} \dots$$

$$= e^{-\lambda}\lambda\left[1 + 2\lambda + \frac{3\lambda^2}{2} \dots\right]$$

$$= e^{-\lambda}\lambda\left[1-\lambda\right]^{-2}.$$

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$$E(x^2) = E[x(x-1)+x]$$
$$= E\{x(x-1)\} + E(x)$$
$$\lambda^2 + \lambda.$$

$$E(x(x-1)) = \cancel{E(x) + E(x-1)}.$$
$$= \sum_{x=1}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{(x\cancel{\infty})!}$$

$$= e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^x}{(x-2)!}$$

$$= e^{-\lambda}\left[\lambda^2 + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots\right]$$

$$= \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$
$$= \lambda^2.$$

$$E(x^2) = \lambda^2 + \lambda.$$

$$\text{Var } X = E(x^2) - [E(x)]^2$$
$$= \lambda^2 + \lambda - \lambda^2$$
$$= \lambda.$$

Recurrence Relation for moments of Poisson dist^r.

Sol^n $X \sim P(\lambda)$

$$\mu_r = E[(x - \underbrace{E(x)}_{\text{Mean}})^r] = \sum_{x=0}^{\infty}(x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!} \quad -(1)$$

diff wrt λ

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} -r(x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty}\frac{(x-\lambda)^r}{x!}\left(e^{-\lambda}x\lambda^{x-1} - e^{-\lambda}\lambda^x\right)$$

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$$\frac{d\mu_r}{d\lambda} = -r\sum_{x=0}^{\infty}\frac{(x-\lambda)^{r-1}e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty}\frac{(x-\lambda)^{r}}{x!}e^{-\lambda}\lambda^{x-1}(x-\lambda)$$

$$= \sum_{x=0}^{\infty}\cancel{(x-\lambda)^{x-1}}e^{-\lambda}\lambda^{x-1}$$

$$= -r\mu_{r-1} + \frac{1}{\lambda}\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{x!}(x-\lambda)^{r+1}$$

$$= -r\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$$\frac{d\mu_r}{d\lambda} = -r\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

Thus $\lambda\left(\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right) = \mu_{r+1}$

* $\mu_0 = 1$; in (1)
* pmf of poisson = 1.

$$\mu_1 = \sum_{x=0}^{\infty}(x-\lambda)^{1}\frac{e^{-\lambda}\lambda^{x}}{x!} = E\left[(X-E(X))\right]$$

$$= E\left[X-\lambda\right]$$

$$= E(\lambda)-\lambda$$

$$= \lambda-\lambda$$

$$= 0.$$

2nd central moment $\mu_2 = \left(\lambda\left(\frac{d\mu_1}{d\lambda} + \mu_0\right)\right) = \lambda.$ μ_4

$$\mu_3 = \lambda\frac{d\mu_2}{d\lambda} + \mu_2 = \lambda.$$

$$\mu_4 = 3\lambda^2+\lambda$$

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mgf of poisson distr

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{xt}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t-1)}$$

Characteristic f^n

$$M_x(t) = E[e^{itx}] = \cancel{e^{\lambda}(e} \; e^{\lambda(e^{it}-1)}$$

~~Additive Prop~~

* $$K_x(t) = \log M_x(t) = \log e^{\lambda(e^t-1)}$$

$$= \lambda(e^t-1)$$

$$= \lambda\left[\cancel{1} + t + \frac{t^2}{2} + \cdots \cancel{-1}\right]$$

$$= \lambda t\left[1 + \frac{t}{2} + \cdots\right]$$

Prove *

K_r = coefficient of $\frac{t^r}{r!}$ in $K_x(t)$

$K_1 = \lambda$

$K_2 = \lambda$

$K_r = \lambda \qquad r = 1, 2, \ldots$

Normal – Symmetric
Other – Skew symmetric.

Q β_1 (Skewness) $= \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$; $\gamma_1 = \sqrt{\beta_1}$

Kurtosis.

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2+\lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$; $\gamma_2 = \beta_2 - 3$

$\lambda \to \infty$, $\beta_1 \to 0$ Normal Dist^r.
$\beta_2 \to 3$.

Q In PD if m is mean σ is SD. Prove $m\sigma\gamma_1\gamma_2 = 1$.

Sol^n ~~$\lambda\lambda \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda} = 1$~~. $m = \lambda$

$\sigma = \sqrt{\lambda}$

$\gamma_1 = \frac{1}{\sqrt{\lambda}}$

$\gamma_2 = \frac{1}{\lambda}$.

$m\sigma\gamma_1\gamma_2 = \lambda \cdot \sqrt{\lambda} \cdot \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\lambda} = 1$. H.P.

Additive Property of Independent Poisson Variate:

* Sum of independent Poisson variate is also Poisson variate

Theorem If X_i $(1,2,\dots,n)$ are ind. poisson variate with parameter λ_i then ΣX_i is also Poisson variate with parameter $\Sigma\lambda_i$

Proof Let $X_i \sim P(\lambda_i)$ be ind. poisson

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$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) + M_{X_2}(t) + \cdots + M_{X_n}(t)$$

$(\because X_i$'s are ind.$)$

$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \cdots$$

$$= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)}$$

$$= e^{(\sum_{i=1}^{n}\lambda_i)(e^t-1)}$$

Q Show that in PD with unit mean. Mean deviation about mean is $\left(\frac{2}{e}\right)$ times S.D.

Ans ~~Mean = λ = 1.~~ Given: $X \sim P(1)$

~~$E[X-1] = \frac{2}{e}$~~

M.D. about $= E[|X-\text{mean}|]$

$= E[|X-1|]$

$$= \sum_{x=0}^{\infty} |(x-1)| \frac{e^{-1}\, 1^x}{x!}$$

$\frac{4-1\, e^{-1}}{4!}$

$$= \frac{|-1|e^{-1}}{0!} + \frac{|2-1|e^{-1}}{2!} + \frac{|3-1|e^{-1}}{3!}$$

$$= \frac{1}{e} + \frac{1}{2e} + \frac{1}{3e^2} + \cdots \quad + \frac{1}{e} - \frac{1}{e}$$

$$= \frac{1}{e}\left[1 + \frac{1}{2} + \frac{1}{2}\right.$$

$$= \frac{1}{e} + \frac{1}{e}\left[\frac{1+1}{2} + \right.$$

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MD about mean $= e^{-1} \sum_{x=0}^{\infty} \frac{|x-1|}{x!}$

$= e^{-1}\left[1 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots\right].$

General term $\frac{n}{(n+1)!} = \frac{n+1-1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}$

$= \frac{1}{n!} - \frac{1}{(n+1)!}$

$= e^{-1}\left[1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots\right]$

$= \frac{2}{e}.$

Q 2% of items made by machine are defective find prob. that 3 or more items are defective in sample of 100 items. Given $e^{-2} = 0.135$

Soln $\lambda = np = 100\left(\frac{2}{100}\right) = 2.$ $\boxed{\lambda = 2}$ $P(x,\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$

$P(X \geq 3) = 1 - P(X < 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right]$

$= 1 - e^{-2}[1 + 2 + 2]$

$= 1 - 5 \times 0.135$

$= 1 - 0.675$

$= 0.325$

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When prob. is n is large (more than 20)
$p = 0.05$ (very small) 0.1 acceptable

Q A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pin will defective. What is approx. prob. that box will fail to meet the guaranteed quality?

Sol^n $n = 100$, $p = \frac{5}{100}$ $\lambda = np = 5$

$P(X > 10) = 1 - P(X \le 10)$ x = no of defective cotter pin

$= 1 - \sum_{x=0}^{10} \frac{e^{-5} 5^x}{x!}$

Q If X and Y are ind. Poisson Variates. Show conditional distribution of X. Given X+Y is binomial

Proof Let X and Y be independent Poisson Variate with parameter, $\lambda + \mu$

$$P\left(X=r \,/\, X+Y=n\right) = \frac{P[X=r \cap X+Y=n]}{P[X+Y=n]}$$

$$= \frac{P(X=r \cap Y=n-r)}{P(X+Y=n)}$$

$$= \frac{P(X=r)\, P(Y=n-r)}{P(X+Y=n)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^r}{r!} \frac{e^{-\mu}\mu^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}} = \frac{(\lambda^r \mu^{n-r})\, n!}{r!\,(n-r)!\,(\lambda

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$$P\left(X=r \,\middle|\, X+Y=n\right) = {}^{n}C_r \left(\frac{\lambda}{\lambda+\mu}\right)^{r} \left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

$$= {}^{n}C_r \cdot p^{r} q^{n-r}$$

$$p = \frac{\lambda}{\lambda+\mu} \qquad q = \frac{\mu}{\lambda+\mu} \; (1-p)$$

i.e. cond^n dist^r of X+Y is binomial.

Q1 If X is poison variate with mean m. show $\frac{X-m}{\sqrt{m}}$ is variable with mean zero and variance unity. find m.g.f. for the variable and show it approaches $e^{t^2/2}$ (standard Normal variate ; Mean = 0, Variance = 1) as $m \to \infty$

Q2 If X is poison variate with parameter m, μ_r is r^{th} central moment Prove

R

$$m\left[{}^{r}C_1 \mu_{r-1} + {}^{r}C_2 \mu_{r-2} + \cdots + {}^{r}C_r \mu_0\right] = \mu_{r+1}.$$

Hint $\mu_{r+1} = E\left[X - E(X)\right]^{r+1}$

$$= E\left[X-m\right]^{r+1}$$

$$= \sum_{x=0}^{\infty} (x-m)^{r+1} \frac{e^{-m} m^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} (x-m)^{r} \frac{(x-m) e^{-m} m^{x}}{x!}$$

$y = x-1$

$$= \sum_{x=1}^{\infty} \frac{(x-m)^{r} e^{-m} m^{x}}{(x-1)!} - m\sum_{x=0}^{\infty} \frac{(x-m)^{r} e^{-m} m^{x}}{x!}$$

$$= m\sum_{y=0}^{\infty} \frac{(y-m+1)^{r} e^{-m} m^{y}}{}$$

$$= m\sum_{y=0} {}^{r}C_0 (y-m)^{r} 1^{0} + {}^{r}C_1 (y-m)^{r-1} 1$$

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$$= \sum_{y=0}^{\infty} \left[{}^{r}C_0 (y-m)^r + {}^{r}C_1 (y-m)^{r-1} + {}^{r}C_2 (y-m)^{r-2} + \dots\right] \frac{e^{-m} m^{y}}{y!}$$

$$\mu_{r+1} = \sum_{y=0}^{\infty} (y-m)^r \frac{e^{-m} m^{y+1}}{y!} - m\mu_r + {}^{r}C_1 \mu_{r-1} + {}^{r}C_2 \mu_{r-2} + \dots$$

$$\mu_{r+1} = m\mu_r - m\mu_r + {}^{r}C_1 \mu_{r-1} + {}^{r}C_2 \mu_{r-2} + \dots$$

$$\mu_{r+1} = {}^{r}C_1 \mu_{r-1} + {}^{r}C_2 \mu_{r-2} + \dots {}^{r}C_0 \mu_0$$

Sol^n (i) $Y = \frac{X-m}{\sqrt{m}}$ $E(Y) = \frac{1}{\sqrt{m}} E(X-m) = 0$

$V(Y) = E\left(\frac{X-m}{\sqrt{m}}\right)^2 = \frac{1}{m} E(X-m)^2$

$= \frac{1}{m} \mu_2 = 1$ $E(x^2 - 2mX + m^2)$

$E(x^2) - m^2$

$E(x^2) = m^2 + m - m^2 = 1$

(ii)

$Y = \frac{X-m}{\sqrt{m}}$ has zero mean and variance unity

MGF of $Y = M_Y(t) = E(e^{tY})$

$= E\left[e^{t(X-m)/\sqrt{m}}\right]$

$= e^{-\sqrt{m}t} \left[E\left(e^{tX/\sqrt{m}}\right)\right]$

$= e^{-\sqrt{m}t} \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} e^{tx/\sqrt{m}}$

$= e^{-\sqrt{m}t} \cdot e^{-m} \sum_{x=0}^{\infty} \frac{(m e^{t/\sqrt{m}})^x}{x!}$

$= e^{-m-\sqrt{m}t} \left[1 + \frac{m e^{t/\sqrt{m}}}{1!} + \frac{(m e^{t/\sqrt{m}})^2}{2!} + \dots\right]$

$= e^{-m-\sqrt{m}t} \, e^{m e^{t/\sqrt{m}}}$

$= \exp\left\{-m - \sqrt{m}t + m\left(1 + \frac{t}{\sqrt{m}} + \frac{t^2}{2!\,m} + \dots\right)\right.$

$= \exp\left\{\frac{t^2}{2} + \frac{1}{3!}\frac{t^3}{\sqrt{m}}\right.$

Now proceeding to limit as $m \to \infty$

$$\lim_{m\to\infty} M_Y(t) = \exp\left(\frac{t^2}{2}\right)$$