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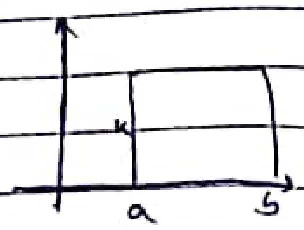
UNIT – IV.

Continous Distribution

Rectangular (Uniform Distribution)

A r.v X is said to have continous rect. Dist^n in $[a,b]$ if its pdf is given by



① • p.d.f

$$\int_a^b f(x)\,dx = 1$$

$$\Rightarrow \int_a^b k\,dx = 1.$$

$$\sin h(at) = at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \dots$$

$$\Rightarrow k(b-a) = 1$$

$$\Rightarrow k = \frac{1}{b-a}.$$

$$f(x) = \begin{cases} \dfrac{1}{b-a} & ; a \le x \le b \\ 0 & , \text{otherwise.} \end{cases}$$

② • Moments of $X \sim U(a,b)$

$$\mu_r' = E(X^r) = \int_a^b x^r f(x)\,dx$$

$$\Rightarrow \mu_r' = \int_a^b x^r \cdot \left(\frac{1}{b-a}\right) dx.$$

$$\Rightarrow \mu_r' = \left(\frac{1}{b-a}\right) \left| \frac{x^{r+1}}{r+1} \right|_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^{r+1} - a^{r+1}}{r+1} \right]$$

Put $r = 1$

$$\mu_1' = \frac{b+a}{2}.$$

μ_r' is coefficient of $\frac{t^r}{r!}$ in $M_x(t)$

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$$\mu_2' = \frac{1}{(b-a)} \frac{b^3-a^3}{3}$$
$$= \frac{1}{3}(b^2+a^2+ab)$$

Variance $\mu_2 = \mu_2' - (\mu_1')^2$
$$= \frac{1}{3}(b^2+a^2+ab) - \frac{(a+b)^2}{4}$$
$$= \frac{1}{12}\left[4b^2+4a^2+4ab-3a^2-3b^2-6ab\right]$$
$$= \frac{1}{12}\left[b^2+a^2-2ab\right] = \frac{1}{12}(a-b)^2.$$

(3) mgf of Uniform Distr. $X \sim U[a,b]$

$$MGF = E[e^{tx}]$$
$$= \int_a^b e^{tx} f(x)\,dx.$$
$$= \frac{1}{(b-a)} \int_a^b e^{tx}\,dx = \frac{1}{(b-a)} \frac{e^{tb}-e^{ta}}{t}.$$
$$M_x(t) = \frac{1}{(b-a)t}(e^{bt}-e^{at})$$

Q $f(x) = \frac{1}{2a}$, $-a \le x \le a$

Show mgf is $\frac{1}{at}\sinh at$. find Mean and Variance, μ_{2n}

Sol^n $MGF = M_x(t) = E[e^{tx}]$
$$= \int_{-a}^{a} e^{tx} \cdot \frac{1}{2a}\,dx$$
$$= \frac{1}{2at}\left[e^{ta}-e^{-ta}\right] = \frac{1}{2at} \sinh at$$

18)

Mean

$\mu_1 = \frac{-a+a}{2} = 0$

$\mu_2 = \frac{1}{12}(a+a)^2 = \frac{a^2}{3}$ $[\mu_2 = \mu_2' - (\mu_1)^2]$

$[\because \mu_2' = \frac{1}{3}(a^2+a^2-a^2) = \frac{a^2}{3}]$

μ_{2m}

Note $\boxed{\begin{array}{l} \text{Mean} = 0 \\ \mu_r = \mu_r' \end{array}}$

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Q2 If X is uniform distribution with mean 1 and variance $\frac{4}{3}$. Find probability $P(x<0)$

Sol^n $X \sim U[a,b]$

Mean of $X = 1 \Rightarrow \frac{b+a}{2} = 1 \Rightarrow b+a = 2$

$Var(X) = \frac{4}{3} \Rightarrow \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow (b-a) = \pm 4.$

Taking $b-a = 4$
$a = 3$ $b = -1$

$b-a = -4.$
$a = -1$ $b = 3.$

But $a<b$ $\therefore$ $a = -1$, $b = 3$ is considered.

$$P(X<0) = \int_{-1}^{0} f(x)dx =$$

$$= \int_{-1}^{0} \frac{1}{4} \cdot dx = \frac{1}{4}.$$

Q3 Let $X \sim U[0,1]$ with pdf $f(x) = 1$ $0 \le x \le 1$ and 0. Find pdf for $-2\log x$.

~~pdf =~~ $\cancel{\int_{0}^{1} 1 \cdot dx} =$

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Note Distributive function (cdf) $= P(\alpha \le x)$

pdf :- $\boxed{f(x) = \frac{dF}{dx}}$

Sol^n Let $Y = -2\log X$.

$G(y) = P(Y \le y) = P(-2\log X \le y)$

$\Rightarrow G(y) = P(2\log X \ge -y)$

$\Rightarrow G(y) = P(\log X \ge -\frac{y}{2})$

$= P(X \ge e^{-y/2})$.

$= 1 - P(X \le e^{-y/2})$.

$= 1 - \int_0^{e^{-y/2}} dx$

$= 1 - e^{-y/2}$.

$g(y) = \frac{d}{dy} G(y) = \frac{1}{2} e^{-y/2}$.

S/t is called as exponential distribution $\theta = 1/2$

pdf $= \theta e^{-\theta y}$.

Imp **Normal Distribution.** Random variable X is said to follow distribution if its pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0. \end{array}$$

$X \sim N(\mu, \sigma^2)$.

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Mode : Value of x for which $f(x)$ [p.d.f] is max

i.e. mode is sol^n of $f'(x)=0$ and $f''(x)<0$

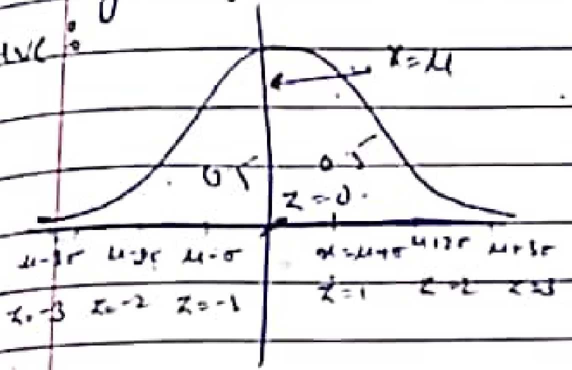
Let $z = \frac{X-\mu}{\sigma}$ is called Standard normal Variable

~~which~~ with mean 0 and Variance 1.

mgf of Standard Normal Variance = $\exp(t^2/2)$

Curve :

as $n \to \infty$



($\because$ it is symmetric curve so area divides uniformly)

Chief Characteristic of Normal prob curve.

i) It is bell shaped with symmetric about $x=\mu$.

ii) Mean Mode and Variance are same for normal distribution

iii) As x ↑es, $f(x)$ ↓es. max value of $f(x)$ ~~at $x=\mu$~~

at $x=\mu$ $\quad (f(x))_{x=\mu} = \frac{1}{\sqrt{2\pi}\,\sigma}$

iv) Since, being the probability can never be negative. no portion of curve lie below x-axis (x-axis is asymptote)

v) Linear combination of ind. Normal variate is also Normal variate i.e.

If $X_1, X_2, \dots, X_n$ are normal variate

Uniqueness Value of z. at 1.

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then $c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$ is also normal variate

vi) Mean deviation about Mean $= \sqrt{\frac{2}{\pi}}\, \sigma \approx \frac{4}{5}\sigma$.

~~Table done~~

vii) Area property ① $P(\mu - \sigma < X < \mu + \sigma)$

$Z = \frac{X-\mu}{\sigma}$ ∴ $P(-1 < Z < 1) = P(-1<Z<0) + P(0<Z<1)$

$= P(0<Z<1) + P(0<Z<1)$

$= 2P(0<Z<1)$

$= 0.6826$

② $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$.

Imp ③ $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$.

Mean and Variance $= \mu, \sigma^2$.

MGF of Normal Distribution.

$X \sim N(\mu, \sigma^2)$, then pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right].$$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

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$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{xt} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma}$. $\quad dx = \sigma dz$.

$$M_x(t) = \frac{1}{\sqrt{2\pi}\cancel{\sigma}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-z^2/2} \cancel{\sigma} dz.$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z)} dz.$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z + \sigma^2 t^2)} e^{\frac{1}{2}\sigma^2 t^2}.$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}(z-\sigma t)^2 - \frac{1}{2}\sigma^2 t^2\right]} dz.$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}.} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

let $z - \sigma t = u$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du.$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}} \sqrt{2\pi}}{\sqrt{2\pi}.}$$

$$= e^{\mu t + \sigma^2 t^2/2}.$$

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$MGF = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$

Standard Normal $\mu = 0$ $\sigma = 1$. $MGF = \exp(t^2/2)$

Cumulative ~~Distribut~~ Generating f^n of Normal

$K_x(t) = \log M_x(t) = \log e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$= \mu t + \frac{\sigma^2 t^2}{2}$.

$\mu_1' = \text{Mean}(k_1) = \text{Coefficient of } t \text{ in } k_x(t)$
$= \mu$

$\mu_2 \quad k_2 = \text{Coefficient of } \frac{t^2}{2!} \text{ in } k_x(t)$
$= \sigma^2$.

Previous done

$\mu_3 = k_3 = 0$.

μ_4

$$\boxed{\mu_4 = K_4 + 3K_2^2}$$

$\mu_4 = 0 + 3\sigma^4$.

$$\boxed{\mu_4 = 3\sigma^4}$$

① Mode of Normal Dist^n

Consider ~~Standard~~ Normal dist^r

$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\log f(x) = \log c - \frac{(x-\mu)^2}{2\sigma^2}$

$c = \frac{1}{\sqrt{2\pi}\sigma}$.

Stats

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diff wrt x.

$$\frac{1}{f(x)} f'(x) = \frac{-1}{2\sigma^2} \cdot 2 (x - \mu)$$

$$f'(x) = \frac{-1}{\sigma^2} (x-\mu) f(x)$$

$$f''(x) = -\left[\frac{1}{\sigma^2} f(x) + \frac{1}{\sigma^2}(x-\mu) f'(x)\right]$$

$$= \frac{-f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right].$$

$f'(x) = 0 \Rightarrow x = \mu$

$f''(x) < 0$ $\quad \mu$

Thus $x = \mu$ is Mode of Normal distr

(2) Median : If M is median of of Normal distr.

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} \Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = \frac{1}{2}.$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \frac{1}{2}$$

$$\text{But } \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx =$$

$$\frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-z^2}{2}\right\} dz = \frac{1}{2}.$$

$$\Rightarrow \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx = \frac{1}{2}.$$

$\Downarrow$ 0

$\Rightarrow \mu = M$

Mean Deviation from Mean

MD(about mean) $= \int_{-\infty}^{\infty} |x-\mu| f(x) \, dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu| e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = z$

$dx = \sigma dz$

$$MD = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| e^{-z^2/2} \sigma \, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz \qquad \because \sigma \text{ is always } +ve$$

$\because$ integrand $|z| e^{-z^2/2}$ is even

$$MD = \sqrt{\frac{2}{\pi}} \sigma \int_{0}^{\infty} z e^{-z^2/2} dz$$

$$= \sqrt{\frac{2}{\pi}} \sigma \int_{0}^{\infty} e^{-t} dt$$

$$= \sqrt{\frac{2}{\pi}} \sigma \qquad = \sqrt{\frac{2 \cdot 7}{22}} = \sqrt{\frac{14}{22}}$$

$$\approx \frac{4}{5} \sigma$$

Obtain recurrence formula for Normal Dist^r.

$$\mu_{2n} = (2n-1) \sigma^2 \mu_{2n-2}$$

Proof

$$\mu_{2n} = E\left[(X-\mu)^{2n}\right] = \int_{-\infty}^{\infty} (x-\mu)^{2n} \cdot f(x) \, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} \exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

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Put $\frac{x-\mu}{\sigma} = z$

$dx = \sigma dz$

$$\mu_{2n} = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n} e^{-z^2/2} \cancel{\sigma} dz$$ [All odd integrands being zero because it become odd f^n]

$$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_{0}^{\infty} z^{2n} e^{-z^2/2} dz.$$

Put $\frac{z^2}{2} = t.$

$2z\,dz = dt$

always

$$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_{0}^{\infty} (2t)^{n-1/2} e^{-t}\, dt \quad \cancel{(2t)^{1/2}}$$

$$= \frac{\cancel{2}\sigma^{2n} 2^n}{\sqrt{2\pi}} \int_{0}^{\infty} t^{n-1/2} e^{-t}\, dt.$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_{0}^{\infty} t^{n+1/2-1} e^{-t}\, dt. \qquad \Gamma 2n = \Gamma n\, \Gamma n+1/2$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} . \Gamma n+1/2$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} (n-1/2)(n-3/2) \cdots \frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi}$$

$$= \cancel{2^n}\, \sigma^{2n} \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{\cancel{2^n}}$$

Now replacing n by $n-1$.

$2(n-1)-1$
$2n-2-1$

$= \cancel{\sigma^{2n}}$

$$\mu_{2n-2} = \sigma^{2n-2}(2n-3)(2n-5)\cdots 3\cdot 1$$

$$\mu_{2n} = \sigma^2 (2n-1)\,\mu_{2n-2}.$$

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$$\boxed{\mu_{2n} = (2n-1)\sigma^2 \mu_{2n-2}}$$

① $\mu_2 = (2-1)\sigma^2 \mu_{2\cdot1-2}$

$\mu_2 = \sigma^2$

② $\mu_4 = 3\sigma^2\mu_2$

$\mu_4 = 3\sigma^4$

for Normal distⁿ

$\mu_1 = \mu_3 = \mu_5 \cdots\cdots = 0.$

i.e central moment of odd order $\boxed{\mu_{2n+1} = 0}$

A linear combination of ind. Normal variate is also normal variate i.e. if X_i's are ind Normal variate then $\Sigma a_i X_i$ is also Normal Variate

Uniqueness $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots$

let X_i $(i=1,2,\ldots n)$ be n ind. normal variate with mean μ_i and variance σ_i^2.

$M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$

then

$M_{\Sigma a_i x_i}(t) = M_{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$

$= M_{a_1x_1}(t) \cdot M_{a_2x_2}(t) \cdots M_{a_nx_n}(t)$

$= M_{x_1}(a_1t)\, M_{x_2}(a_2t) \cdots M_{x_n}(a_nt)$

[$\because$ all X_i's are independent Normal variate]

Mean can be +ve

$M_{\Sigma a_i x_i}(t) = e^{\mu_1 a_1 t + \frac{1}{2}\sigma_1^2 (a_1 t)^2} \, e^{\mu_2 a_2 t + \frac{1}{2}\sigma_2^2 (a_2 t)^2} \cdots$

$+ e^{\mu_n a_n t + \frac{1}{2}\sigma_n^2 (a_n t)^2}$

$= e^{t(\mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_n a_n)} \, e^{\frac{1}{2} t^2 (\sigma_1^2 a_1^2 + \sigma_2^2 a_2^2 + \cdots \sigma_n^2 a_n^2)}$

$= e^{t(\sum_{i=1}^{n} \mu_i a_i) t} \, e^{\frac{1}{2} t^2 \sum_{i=1}^{n} \sigma_i^2 a_i^2}$

$= e^{(\sum_{i=1}^{n} a_i \mu_i) t + \frac{1}{2} (\sum_{i=1}^{n} a_i^2 \sigma_i^2) t^2}$ — (1)

from (1) mgf of Normal dist^n

* mean $= \Sigma a_i \mu_i$

* Variance $= \sum_{i=1}^{n} a_i^2 \sigma_i^2$

Q $X \sim N(1,4)$

$Y \sim N(2,9)$

What is dist^n of $X+Y$, $X-Y$, $2X+3Y$.

Sol^n $a_1 = a_2 = 1$

$\mu_1 = 1 \quad \sigma_1^2 = 2.$

$\mu_2 = 2 \quad \sigma_2^2 = 3$

$E(X+Y) = E(X) + E(Y)$ $\qquad E(X-Y) = 1-2$

$= 1 + 2$ $\qquad = -1$

$V(X+Y) = V(X) + V(Y)$ $\qquad V(X-Y) = V(X) + V(Y)$

$= 4 + 9$ $\qquad = 13.$

$= 13.$

$E(2X+3Y) = 2 + 6 = 8.$

$V(2X+3Y) = 4V(X) + 9V(Y)$

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Q. X is Normal Variate with mean 30 and sd 5 find

i) $P(26 \le X < 40)$

$X \sim N(\mu, \sigma^2)$

$\mu = 30 \quad \sigma = 5 \therefore \sigma^2 = 25$

i) $X = 26 \quad z = \frac{X-\mu}{\sigma} = \frac{26-30}{5} = \frac{-4}{5} = -0.8$

$X = 40 \quad z = \frac{40-30}{5} = 2$

$P(26 \le X \le 40) = P(-0.8 \le z \le 0) + P(0 \le z \le 2)$

$\cancel{= P(0.8 \ge z) + P(z \le 2)}$

$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$ ∵ symm

$= 0.2881 + 0.4772$

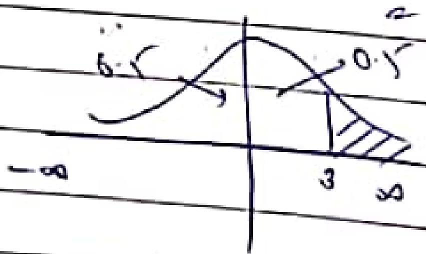
$= 0.7653$

ii) $P(X \ge 45) = P(z \ge 3) = 0.5 - P(0 \le z \le 3)$

$\frac{45-30}{5} = 3$

$= 0.5 - 0.49865$

$= 0.00135$



iii) $P\{|X-30| \le 5\} = P\{25 \le X \le 35\}$

$= P\{-1 \le z \le 1\}$

$= 2P(0 \le z \le 1)$

$= 2 \times 0.3413 = 0.6826$

0 1 2 3 . —

0.00
0.01
0.02

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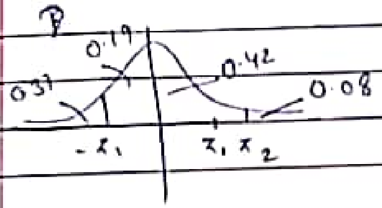
Q In Normal Distⁿ, 31% items are under 45 and 8% over 64. find Mean and variance of distⁿ

Solⁿ Given that $P(X < 45) = \frac{31}{100} = 0.31$

$$P(X > 64) = \frac{8}{100} = 0.08$$



P/ When $X = 45$, $z = \frac{X-\mu}{\sigma} = \frac{45-\mu}{\sigma} = -z_1$

$P(z < -z_1) = 0.31$.

$P(-z_1 < z < 0) = 0.19$.

∴ symmet

$P(0 < z < z_1) = 0.19$ (Symm. prop).

$\Rightarrow z_1 = 0.496$ (from Normal table)

$P(X > 64) = 0.8$ $\quad$ $X = 64$, $z = \frac{X-\mu}{\sigma}$, $\frac{64-\mu}{\sigma} = z_2$.

$P(z > z_2) = 0.08$

$P(0 < z < z_2) = 0.42$

$z_2 = 1.405$ (Normal table)

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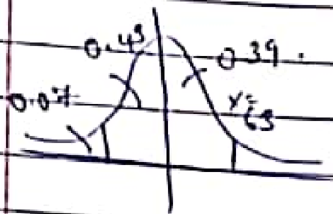
In Normal Dist^n 7% items are under 35 and 89% are under 63. Find mean and variance

Sol^n



$P(X<63) = 0.89$

$P(X<35) = 0.07$

$P(0<X<35) = 0.43$ (By Symmetric)

$P(0<X<63) = 0.39$

0.89
0.50
0.39

$X = 35 \qquad z = \frac{X-\mu}{\sigma} = \frac{35-\mu}{\sigma} = -z_1$

$X = 63 \qquad z = \frac{X-\mu}{\sigma} = \frac{63-\mu}{\sigma} = z_2$

$P(0<Z<z_1) = 0.43$

$z_1 = 1.48$

$P(0<Z<z_2) = 0.39$

$z_2 = 1.23$

$\frac{35-\mu}{\sigma} = -1.48$

$\frac{63-\mu}{\sigma} = 1.23$

$35 = -1.48\sigma + \mu \qquad 35 =$

$63 = 1.23\sigma + \mu$

$28 = 2.71\sigma$

$\sigma = 10.33$

Q: The distribution of weekly wage for 500 workers in factory is approx normal with mean and sd of ₹ 75 and ₹ 15. find no of workers who receive weekly

i) more than Rs 90

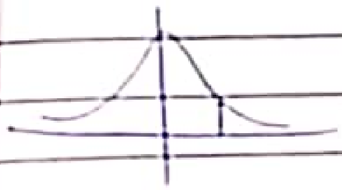
ii) less than Rs 45

Sol: $\mu = 75 \quad \sigma = 15$.

Let X be no of workers which who receive weekly wage.

$P(X > 90) = P(Z > 1)$.

If $x = 90 \qquad \frac{90-75}{15} = 1$



$P(Z > 1) = 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413$ (Normal table)

$= 0.1587$

No of worker $= 500 \times 0.1587$

$= 79.35$

$= 79$ worker

$P(X < 45) = P(Z < -2)$.

$= 0.5 - P(0 < Z < 2)$

$\frac{45-75}{15}$

$= 0.5 - 0.4772$

$= 0.0228$

No of worker $= 500 \times 0.0228$

$= 11.4000 = 11$ worker.

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$\lim_{x\to\infty} e^{x} x^{\lambda-1} = \Gamma\lambda$

find

* Normal $dist^r$ is special case of Binomial $dist^r$
i) $n \to \infty$
ii) Neither p nor q is small

Gamma Dist^r
Continuous rv X which has following pdf

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} & \lambda>0 \quad 0<x<\infty. \\ 0 & \text{otherwise.} \end{cases}$$

Mgf of Gamma Dist^r $\qquad X \sim \gamma(\lambda)$

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\, dx.$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} dx.$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-x(1-t)} x^{\lambda-1}.\, dx$$

$$= \frac{1}{\Gamma\lambda} \frac{\Gamma\lambda}{(1-t)^\lambda}$$

$$= (1-t)^{-\lambda}.$$

$$= 1+\lambda t + \frac{\lambda(\lambda+1)}{2} t^2 + \dots$$

$\lambda^2 - \lambda^2$

Mean $= \lambda$

Variance $= \lambda(\lambda+1) - \lambda^2 \left[\cancel{\mu_2' - (\mu_2')^2} \right] \quad \mu_2 = \mu_2' - (\mu_1')^2$

$\mu_2 \quad = \lambda$

Cumulants of Gamma f^n

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$K_X(t) = \log M_X(t)$
$= \log (1-t)^{-\lambda}$
$= -\lambda \log(1-t)$
$= \lambda t + \frac{\lambda t^2}{2} + \frac{\lambda t^3 \cdot 2}{3! \cdot 2} + \frac{6\lambda t^4}{6 \cdot 4} + \cdots$

$K_1 = \lambda$
$K_2 = \lambda$
$K_3 = 2\lambda$
$K_4 = 6\lambda$

Additive Property
The sum of independent Gamma Variate is also Gamma Variate i.e. if $X_1, X_2, \ldots, X_k$ are k independent Gamma Variate with parameters $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_k$ then
$X_1 + X_2 + \cdots + X_k$ is also Gamma Variate with parameter $\sum_{i=1}^{k} \lambda_i$

$M_{X_1+X_2+\cdots X_k}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_k}(t)$
[$\because$ ~~X_i's are Indep. Ind.~~]

$= (1-t)^{-\lambda_1} (1-t)^{-\lambda_2} \cdots (1-t)^{-\lambda_k}$
$= (1-t)^{-(\lambda_1+\lambda_2 \cdots + \lambda_k)}$

which is mgf of Gamma Variate with the ~~diameter~~ parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_k$

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}dx = B(\mu,\nu)$$

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Beta distribution of first kind

$$f(x) = \begin{cases} \dfrac{1}{B(\mu,\nu)} x^{\mu-1}(1-x)^{\nu-1} & ; (\mu,\nu)>0,\ 0<x<1 \\ 0 & \text{Otherwise} \end{cases}$$

where $B(\mu,\nu)$ is Beta f^n of first kind.

Constants of Beta distribution of first kind

$$\mu_r' = E(x^r) = \int_0^1 x^r f(x)dx$$

$$= \frac{1}{B(\mu,\nu)}\int_0^1 x^{r+\mu-1}(1-x)^{\nu-1}$$

$$= \frac{\dfrac{\Gamma(r+\mu)\,\Gamma(\nu)}{\Gamma(r+\mu+\nu)}}{\dfrac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}} = \frac{\Gamma(\mu+r)\,\Gamma(\mu+\nu)}{\Gamma(\mu+\nu+r)\,\Gamma(\mu)} = \frac{B(\mu,\nu)}{B(\mu,\nu)}$$

$$\mu_1' = \text{Mean} = \frac{\Gamma(\mu+1)\,\Gamma(\mu+\nu)}{\Gamma(\mu+\nu+1)\,\Gamma(\mu)}$$

$$= \frac{\mu}{\mu+\nu}$$

$$\mu_2' = \frac{\Gamma(\mu+2)\,\Gamma(\mu+\nu)}{\Gamma(\mu+\nu+2)\,\Gamma(\mu)} = \frac{(\mu+1)\mu}{(\mu+\nu+1)(\mu+\nu)}$$

$$\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}.$$

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$0<x<1$

$$\mu_2 = \mu_2' - \mu_1'^2.$$

$$= \frac{\mu^2(\mu+1)^2}{(\mu+\nu+1)^2(\mu+\nu)} - \frac{\mu^2}{(\mu+\nu)^2}$$

$$= \frac{\mu}{(\mu+\nu)}\left[\frac{\mu+1}{\mu+\nu+1} - \frac{\mu}{\mu+\nu}\right]$$

$$= \frac{\mu}{(\mu+\nu)}\left[\frac{\mu^2+\mu\nu+\mu+\nu-\mu^2-\mu\nu-\mu}{(\mu+\nu)(\mu+\nu+1)}\right]$$

$$= \frac{\mu\nu}{(\mu+\nu)^2(\mu+\nu+1)}$$

(iv) Harmonic Mean.

$$\frac{1}{H} = \int_0^1 \frac{1}{x} f(x)\, dx.$$

$$\frac{B(\mu+1, \nu)}{B(\mu, \nu)}$$

$$= \int_0^1 \frac{1}{x} \cdot \frac{1}{B(\mu,\nu)} x^{\mu-1}(1-x)^{\nu-1} dx$$

$$= \frac{1}{B(\mu,\nu)} \frac{\Gamma\mu-1\,\Gamma\nu}{\Gamma\mu+\nu-1} = \frac{1}{B(\mu,\nu)} B(\mu-1,\nu)$$

$$= \frac{\Gamma\mu+\nu}{\Gamma\mu\,\cancel{\Gamma\nu}} \frac{\Gamma\mu-1\,\cancel{\Gamma\nu}}{\Gamma\mu+\nu-1}$$

$$= \frac{\mu+\nu-1}{\mu-1}$$

$$H = \frac{\mu-1}{\mu+\nu-1}.$$

$\frac{1}{\beta(u,v)} (1+y)^{u-1} y^{-(u+v)} dy.$

II Beta distribution of second kind.

a) pdf

$$f(x) = \begin{cases} \frac{1}{B(\mu,\nu)} \cdot \frac{x^{\mu-1}}{(1+x)^{\mu+\nu}} & ; \quad 0 < x < \infty \\ 0 & ; \text{ otherwise.} \end{cases}$$

Let $1+x = y$ reduces to 1^{st} kind.

b) Constants of Beta distribution of II kind

$$E(x^r) = \int_0^\infty x^r f(x)dx = \frac{1}{B(\mu,\nu)} \int_0^\infty x^r \frac{x^{\mu-1}}{(1+x)^{\mu+\nu}} dx$$

$$= \frac{1}{B(\mu,\nu)} \int_0^\infty \frac{x^{(\mu+r)-1}}{(1+x)^{\mu+r+\nu-r}} dx.$$

$$= \frac{1}{B(\mu,\nu)} \{ B(\mu+r, \nu-r)$$

$$\left[\because \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n) \right]$$

$$= \frac{\dfrac{\Gamma(\mu+r)\,\Gamma(\nu-r)}{\Gamma(\mu+\nu)}}{\dfrac{\Gamma\mu\,\Gamma\nu}{\Gamma(\mu+\nu)}}$$

$$= \frac{\Gamma(\mu+r)\,\Gamma(\nu-r)}{\Gamma\mu\,\Gamma\nu}$$

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$$\mu_1' = \frac{\mu}{\nu-1}.$$

$$\mu_2' = \frac{\Gamma(\mu+2)\,\Gamma(\nu-2)}{\Gamma(\mu)\,\Gamma(\nu)}$$

$$= \frac{(\mu+1)\mu}{(\nu-1)(\nu-2)}$$

$$\mu_2 = \frac{\mu(\mu+1)}{(\nu-1)(\nu-2)} - \frac{\mu^2}{(\nu-1)^2}$$

$$= \frac{\mu}{(\nu-1)}\left[\frac{\mu+1}{\nu-2} - \frac{\mu}{\nu-1}\right]$$

$$= \frac{\mu}{(\nu-1)^2(\nu-2)}\left[\cancel{\mu\nu} - \mu + \nu - 1 - \cancel{\mu\nu} + 2\mu\right]$$

$$= \frac{\mu\,[\mu+\nu-1]}{(\nu-1)^2(\nu-2)}$$

$$H = \int_0^\infty \frac{1}{x}\,\frac{1}{B(\mu,\nu)}\,\frac{x^{\mu-1}}{(1+x)^{\mu+\nu}}\,dx$$

$$= \int_0^\infty \frac{1}{B(\mu,\nu)}\,\frac{x^{\mu-2}}{(1+x)^{\mu-1+\nu+1}}\,dx$$

$$= \frac{1}{B(\mu,\nu)}\,B(\mu-1,\nu+1) = \frac{\dfrac{\Gamma(\mu-1)\,\Gamma(\nu+1)}{\cancel{\Gamma(\mu+\nu)}}}{\dfrac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}} = \frac{\nu}{\mu-1}$$

$$= \frac{\cancel{\Gamma(\mu-1)\,\Gamma(\mu+\nu)}}{\cancel{\Gamma(2\mu+2\nu-2)}}$$

$$H = \frac{\mu-1}{\nu}$$

Q If X and Y are independent Gamma distr. variate with parameter u, v Show

$U = X + Y$ $\quad Z = \frac{X}{X+Y}$ are independent and

U is $\gamma(u+v)$ and Z is $B(u, v)$ variate

Sol^n $X \sim \gamma(u)$ $\quad Y \sim \gamma(v)$

$f_1(x)dx = \frac{1}{\Gamma u} e^{-x} x^{u-1} dx \quad 0 < x < \infty \quad u > 0$

$f_2(y)dy = \frac{1}{\Gamma v} e^{-y} y^{v-1} dy \quad 0 < y < \infty \quad v > 0$

X and Y are independent distributed, joint prob. differential is given by

$dF(x,y) = f_1(x) f_2(y) dx\, dy$

$= \frac{1}{\Gamma u \Gamma v} e^{-(x+y)} x^{u-1} y^{v-1} dx\, dy$

$x + y = U \quad z = \frac{x}{x+y}$

$x = (x+y) z$

$x = Uz$.

& $y = U - x = U - Uz = U(1-z)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} z & U \\ 1-z & -U \end{vmatrix}$$

$J = -Uz - U(1-z) = -Uz - U + Uz$

$= -U$.

Joint dist^n of U and z is

$dG(U, z) = \frac{1}{\Gamma u \Gamma v} e^{-U} (Uz)^{u-1} [U(1-z)]^{v-1} |J| \, du\, dz$

$dx\, dy = r\, d\theta\, dr$

$r = |J|$

variate

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$$\frac{1}{\Gamma u \, \Gamma v} e^{-U} U^{u+v-1} z^{u-1} (1-z)^{v-1} \, dU \, dz$$

$$B(u,v) = \frac{\Gamma u \, \Gamma v}{\Gamma u+v}$$

$$dG(U,z) = \left(\frac{1}{\Gamma u+v} e^{-u} u^{u+v-1} \, du\right) \left(\frac{1}{B(u,v)} z^{u-1} (1-z)^{v-1} dz\right)$$

$0 < u < \infty$
$0 < z < 1$

$$dG(U,z) = g_1(U) dU \; g_2(z) dz$$

U and z are ind. dist^r
$\therefore U \sim \gamma(u,v)$ $\quad z \sim \beta(u,v)$

Imp Q. If x and y are ind. Gamma variate with parameter u and v.

Show $U = x+y$ $\quad z = \frac{x}{y}$ are ind. and $U \sim \gamma(u+v)$
$z \sim \beta_2(u,v)$

Sol^n $dF(x,y) = \frac{1}{\Gamma u \, \Gamma v} e^{-(x+y)} x^{u-1} y^{v-1} dx\,dy$

$0 < x < \infty \quad u > 0$
$0 < y < \infty \quad v > 0$

Let $x+y = u$, $z = \frac{x}{y}$

$\Rightarrow 1+z = 1+\frac{x}{y} = \frac{U}{y}$

$y = \frac{U}{1+z}$ $\quad x = \frac{Uz}{1+z} = U\left(1-\frac{1}{1+z}\right)$

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$$J = \frac{\partial(x,y)}{\partial(u,z)} = \frac{-u}{(1+z)^2}.$$

As x and y range from 0 to ∞, both u and z range from 0 to ∞. Hence the Joint probability differential of random variables U and Z become

$$dG(u,z) = \frac{1}{\Gamma\mu\,\Gamma\nu} e^{-u} \left(\frac{uz}{1+z}\right)^{\mu-1} \left(\frac{u}{1+z}\right)^{\nu-1} |J|\, du\, dz.$$

$$= \left[\frac{e^{-u} u^{\mu+\nu-1}}{\Gamma(\mu+\nu)} du\right] \left[\frac{1}{B(\mu,\nu)} \cdot \frac{z^{\mu-1}}{(1+z)^{\mu+\nu}}\right]$$

$0 < u < \infty$, $0 < z < \infty$.

$U \sim \gamma(\mu+\nu)$

$Z \sim \beta_2(\mu,\nu)$

Imp Q $X \sim N(\mu,\sigma^2)$, obtain pdf of $U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$.

Sol^n $X \sim N(\mu,\sigma^2)$

$$dP(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put $u = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$

$\Rightarrow \frac{x-\mu}{\sigma} = \sqrt{2u}.$

$$\frac{dx}{\cancel{\sigma}} = \frac{\sigma}{\sqrt{2u}} du$$

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$$dG(u) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-u} \frac{\sigma}{\sqrt{2u}} du$$

$$dG(u) = \frac{1}{2\sqrt{\pi}} e^{-u} u^{-1/2} du$$

$$= \frac{1}{2\,\Gamma(1/2)!} u^{1/2-1} e^{-u} du$$

$$U = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \sim \gamma\left(\frac{1}{2}\right)$$

$$\left[\frac{\int_0^\infty u^{1/2-1} e^{-u} du}{\Gamma(1/2)} = \frac{\Gamma(1/2)}{\Gamma(1/2)} = 1 \rightarrow \text{Pdf} \right]$$

$\boxed{\text{To Do}}$

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