$\sum p_i = 1$ Neccey for distr. Q. key !

Q. K3, K4

23/09/19

Unit - II.

Random Experiment :

An experiment whose outcome can't be known in advance is called as random experiment.

eg i) Tossing a coin
ii) Throw a die

Random Variable (r.v.)

It is real no. X connected with outcome of an experiment

eg If E consist of two tosses. r.v. which is no. of heads.

Outcome :	HH	HT	TH	TT
X :	2	~~2~~ 1	1	0

RV → It is a fn that maps element of sample space into Real no.

$X : S \longrightarrow R$.

$$\begin{tikzcd} & R.V. \arrow[dl] \arrow[dr] & \\ \text{Discrete r.v.} & & \text{Continous r.v.} \end{tikzcd}$$

Discrete r.v.

1) It take integral value
... -2 -1 0 1 2 ...

2) $\sum$

3) E(X) - expectation of random variable

$E(X) = \sum x_i \underline{p(x_i)}$ → Prob. mass fn.
(Mean of X) μ_i

Continous r.v.

1) Takes all value b/w given interval. (0, 1)
$x \in (0, 1)$

2) $\int$

3) $E(X) = \int$ ~~$x_i p f(x_i)$~~
$= \int x f(x) dx$

$p(x_i)$ = Prob. Mass f^n (PMF)

i) $p(x_i) \geq 0$

ii) $\sum p(x_i) = 1$.

$f(x)$ = Prob. density

i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$E(x) = \sum x f(x)$ (for discrete r.v.)

$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$ (for continous r.v.)

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$.

Variance of X :

It is defined as

$V(x) = E(x^2) - [E(x)]^2$

(μ_2)

Properties of Expectation

Expectation : The avg value of random phenomenon (mean)

i) Addition theorem of Expectation

If X and Y are two ind. r.v. then

$E(X+Y) = E(X) + E(Y)$

ii) Multiplication theorem of Expectation

If X and Y are two ~~r.v.~~ Independent r.V

$E(XY) = E(X) \cdot E(Y)$

iii) $E(aX+b) = aE(X)+b$.

If $a=1$

$E(X+b) = E(X)+b$.

If $b=0$.

$E(aX+0) = aE(X)$.

Marginal p.d.f of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

where $f_{XY}(x,y)$ is joint p.d.f for X and Y.

Marginal p.d.f of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx.$$

If X and Y are independent r.v. then

$$f_{XY}(x,y) = f_X(x).f_Y(y)$$

Proof of addⁿ theorem

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\; f_{XY}(x,y)\,dx\,dy$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{xy}(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{xy}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{XY}(y)\, dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\right]$$

$$\Rightarrow E(X+Y) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx + \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$\Rightarrow E(X+Y) = E(X) + E(Y)$$

also if $X_1, X_2 - - - -, X_n$ are r.v. then

$$E(X_1 + X_2 ---- + X_n) = E(X_1) + E(X_2) + - - - - - - + E(X_n)$$

ii) Proof for Multiplication theorem

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$ [as X and Y are ind r.v]

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E(X) \cdot E(Y)$$

iii) $E(aX+b) = aE(X) + b$

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= aE(X) + b$$

* If $b = 0$

$E(aX) = aE(X)$.

* $a = 1 \quad b = -\bar{X} = -E(X)$

$E(X - \bar{X}) = 0$.

NA Q1. ~~i)~~ Find expectation of no. on a die thrown.

NA ~~ii)~~ Two unbiased die are thrown of sum of no of pts on them

Sol

x	1	2	3	4	5	6
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = \sum x f(x)$$

$$= (1+2+3+4+5+6)\frac{1}{6}$$

$$= \frac{21}{6} = \frac{7}{2}.$$

Q3 If $E(X) = 3.$

NR $E(2X) = 6. = 2E(x)$

Q4 Let X be the r.v. with probability distribution

NR

x :	-3	6	9
$p(x)$:	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

find $E(x)$ and $V(X)$ $\therefore E(2X+1]^2$

Sol^n

$$E(X) = \frac{-3}{6} + \frac{6}{2} + \frac{9}{3}$$

$$= \frac{-1}{2} + 3 + 3$$

$$= \frac{-1}{2} + 6 = \frac{11}{2}$$

$$E(x^2) = \frac{9}{6} + \frac{36}{2} + \frac{81}{3} = \frac{3}{2} + 18 + 27$$

$$= \frac{3}{2} + 45$$

$$= \frac{93}{2}$$

$\frac{2 \cdot 93}{2 \cdot 2} - \frac{121}{4}$

$$\begin{array}{r} 186 \\ 121 \\ \hline 65 \end{array}$$

$$Var = E(X^2) - (E(X))^2$$

$$= \frac{2 \cdot 93}{2 \cdot 2} - \frac{121}{4} = \frac{\cancel{65}}{\cancel{4}} \cdot \frac{65}{4}.$$

$$\begin{array}{r} 186 \\ 121 \\ \hline 65 \end{array}$$

$$E[4X^2 + 4X + 1] = 4E(x^2) + 4E(x) + E1$$

$$= \cancel{K2}\ 186 + 22 + 1$$

$$= 209$$

$\frac{93}{2}$

Q3 If $E(X)=3$.

NR $E(2X)= 6. = 2E(x)$

Q4 Let X be the r.v. with probability distribution

NR

x :	-3	6	9
$p(x)$:	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$.

find $E(x)$ and $v(x)$. $E(2X+1)^2$

Sol^4 $E(X) = \frac{-3}{6} + \frac{6}{2} + \frac{9}{3}$

$= \frac{-1}{2} + 3 + 3$

$= \frac{-1}{2} + 6 = \frac{11}{2}$

$E(x^2) = \frac{9}{6} + \frac{36}{2} + \frac{81}{3} = \frac{3}{2} + 18 + 27$

$= \frac{3}{2} + 45$

$= \frac{93}{2}$

$\frac{2.93}{2.2} - \frac{121}{4}$

186
121
65

Var $= E(X^2) - (E(X))^2$

$= \frac{2.93}{2.2} - \frac{121}{4} - \cancel{\frac{68}{4}}\ \frac{65}{4}$. 2

186
121
~~6~~5

$E[4X^2+4X+1] = 4E(x^2) + 4E(x) + E1$

$= ~~K2~~\ 186 + 22 + 1$

$= 209$

$\frac{93}{2}$

PAGE NO. :
DATE : / /

Q5 find $E(X)$ if

$$p(x) = \left(\frac{1}{2}\right)^x \quad x = 1, 2, ---.$$

Sol^n

x	1	2	3	$---$
$p(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$- -$

$$E(X) = \frac{1}{2} + 2\cdot\left(\frac{1}{2}\right)^2 + 3\cdot\left(\frac{1}{2}\right)^3 + 4.$$

$$S = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + \frac{3}{8} + 4\cdot\frac{1}{16}$$

$$\cancel{S = 1 + \frac{3}{8} + \frac{4}{16} + ---.}$$

$$S = \frac{1}{2}\left(1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + ---\right)$$

$$\cancel{\frac{1}{2}S = \frac{1}{4}\left(1 + \right.}$$

$$S = \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + ---\right]$$

$$S = \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-2}.$$

$$S = \frac{1}{2}\left[\frac{1}{2}\right]^{-2}$$

$$S = 2.$$

Q6. A continous r.v has pdf if. find $E(X), V(X)$

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

Sol^n $E(X) = \int_0^\infty 2xe^{-2x} \, dx$

$= 2\int_0^\infty xe^{-2x} \, dx$

$= 2\left(\frac{1}{4}\right)$ $\left[\int_0^\infty e^{-ax} x^{n-1} \, dx = \frac{\Gamma n}{a^n}\right]$

$= \frac{1}{2}$

$E(X^2) = \int_0^\infty 2x^2 e^{-2x} \, dx$

$= 2 . \frac{\Gamma 3}{2^3}$

$= \frac{1}{2}$.

$V(X) = E(X^2) - [E(X)]^2$

$= \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{4}$.

Q7 If t is +ve real no.

R

$p(x) = e^{-t}(1-e^{-t})^{x-1}$ Can represent pdf of rv X.

$x = 1, 2, 3 - —$

Find $E(X), V(X)$

$1+4A+9A^2+\dots = (1+A)(1-A)^{-3}.$

x	1	2	3	.
$p(x)$	$e^{-t}(1-t)$			

Sol^n Let $a=(1-e^{-t})$

$$E(x) = \sum^{\infty} x\,p(x)$$
$$= \sum_{x=1}^{\infty} x e^{-t} a^{x-1}$$

$= 1$

x	1	2	3	.
$p(x)$	e^{-t}	$e^{-t}(1-e^{-t})$		.

i) $E(X) = e^{-t}\left[1+2a+3a^2+\dots\right]$

$S = e^{-t}\left[1+2a+3a^2+\dots\right]$

$E(X) = e^{-t}(1-a)^{-2}$

$= e^{-t}\left[1-(1-e^{-t})\right]^{2}$

$= e^{-t}(e^{-t})$

$= e^{t}$

ii) $E(X^2) = e^{-t}\left[1+2^2a+3^2a^2+\dots\right]$

$S\ E(X^2) = e^{-t}\left[1+4a+9a^2+16a^3+\dots\right]$

$-3aS = e^{-t}\left[-3a-12a^2-27a^3+\dots\right]$

$3a^2S = e^{-t}\left[3a^2+12a^3+\dots\right]$

$-a^3S = e^{-t}\left[-a^3+\dots\right]$

$(S-a)^3 = e^{-t}[1+a]$

$(1-a)^3 S = e^{-t}[1+a]$

$S = e^{-t}[1+a)(1-a)^{-3} = e^{-t}\left[2-e^{-t}\right]\left[e^{-t}\right]^{-1}$

$$V(X) = E(X^2) - [E(X)]^2$$
$$= e^{-t}(1+a)(1-a)^{-3} - e^{2t}.$$
$$= \cancel{e^{+t}}\, e^{-t}[2-e^{-t}][e^{-t}]^{-3}.$$

$$= \frac{2-e^{-t}}{e^{-2t}} - e^{+2t}$$
$$= e^{2t}(2-e^{-t}-1)$$
$$= e^{2t}(1-e^{-t})$$

for
- $p(x) \geq 0$
- $\sum p(x) = 1.$

$\frac{1}{e^t} > 0$

$e^t > 1$

a) $p = e^{-t}(1-e^{-t})^{x-1}.$

$e^{-t} > 0$

$e^{t} > 1$

$e^{-t} < 1..$

$\Rightarrow 1-e^{-t} \geq 0$ Thus probability mass f^n..

b) $\sum p(x) = e^{-t}(1-e^{-t})^0 + e^{-t}(1-e^{-t})^1 + \ldots.$
$$= e^{-t}[1+(1-e^{-t}) + (1-e^{-t})^2 + \ldots]$$
$$= e^{-t}\, [1-(1-e^{-t})]^{-1}$$
$\cancel{(1-e^{-t})}$
$$= e^{-t}[e^{-t}]^{-1}$$
$$= 1$$

Q6. A r.v. X can assume positive integral value n with probability proportional to $1/3^n$. Find E(X)

$$P(X=n) \propto 1/3^n,$$

Sol

x	1	2	3	4	---.
$p(x)$	$k/3$	$k/3^2$	$k/3^3$	$k/3^4$	---.

$p(n) \propto \frac{1}{3^n}.$

$\sum p(x) = 1.$ $\frac{k}{3}\left[1+\frac{1}{3}+\frac{1}{3^2}+\ldots\right] = 1$ $\frac{k}{3}\frac{1}{1-1/3} = 1$ $\boxed{k=2}$

$E(x) = \sum x p(x)$

$S = \frac{1}{3}k + \frac{2}{3^2}k + \frac{3}{3^3}k + \cdots$

$\frac{4}{8} S = \cancel{\frac{1}{3}\left[1+\right.} \frac{1}{3}\left[1+\frac{2}{3}+\frac{3}{3^2}+\cdots\right]k.$

$= \frac{2}{3}\frac{1}{(1-1/3)^2} = \frac{4}{2}\cdot\frac{3}{4}\times 2 = \frac{3}{2}$

$(1-\frac{1}{3})^{-2}$ Q7. What is expectation of no of failures preceding the first success in an infinite series of ind. trials with constant prob. p of success in each trial.

Solⁿ Let X be denote no of failures preceding first success

$X = 0, 1, \ldots$

$p(x) = (q \cdot q \ldots x \text{ times}) p = q^x p$

where p = prob. of success

q = prob of failure.

Same as Q2.

Q8 A box contain 'a' white and 'b' black balls 'c' balls are drawn find the expected value of no of white ball drawn.

Solⁿ as $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ ball is white} \\ 0 & \text{if } i^{th} \text{ ball is black} \end{cases}$

∴ c balls are drawn from the box ~~therefore~~

let s be

$s = x_1 + x_2 + \cdots + x_c$

$X_s = a + ba + b^2a + \cdots + b a$

$E(s) = E(x_1 + x_2 + \cdots + x_c)$

$= E\left(\sum_{i=1}^{c} x_i\right)$

$= \sum_{i=1}^{c} E(x_i)$ [By addⁿ theorem]

$E(x_i) = \sum x\, p(x)$

$= 1 p(x=1) + 0 \cdot p(x=0)$

$E(x_i) = 1 \cdot \frac{a}{a+b} = \frac{a}{a+b}$

$E(s) = \sum_{i=1}^{c} \frac{a}{a+b}$

$E(s) = \frac{ca}{a+b}$

VARIANCE :

* Variance $X \Rightarrow E(x^2) - [E(x)]^2$

or

$\underset{\mu_2}{\text{Var } X} = E[(X - E(x))^2]$

* Mean of $X = E(X) = \mu_1' / \mu$.

* $\mu_r' = E(x^r)$

$\mu_1' = E(x)$

$\mu_2' = E(x^2)$

PAGE NO.:
DATE: / /

- $\text{Var } X = E(X^2) - [E(X)]^2$
  $\mu_2 = \mu_2' - (\mu_1')^2$

* Raw Moment $= \mu_r' =$ about origin.
$\mu_r' = E[(X-0)^r]$

* Central Moment $= \mu_r = E[(X-\mu)^r]$
$\downarrow$
$\mu = E(X)$

μ_2 = Variance = 2nd central moment

Properties

i) $V(a) = 0$, a is any constant

ii) $\text{Var}(aX+b) = a^2 \text{Var}(X)$

Proof i) $V(a) = 0$.

$V(x) = E[(x - E(x))^2]$
$V(a) = E[(a - E(a))]$
$= E(a-a)^2$ $[\because E(a) = a]$
$= E(0)$
$= 0$.

ii) $\text{Var}(aX+b) = E[((aX+b) - E(aX+b))^2]$
$= E[[aX - E(aX)]^2]$
$= a^2 E[(X - E(X))^2]$
$= a^2 \text{var } X$.

Ans
Q.eg $\text{Var } X = 3.$

$\text{Var}(2X) = 12$

R If X and Y are ind.

$\text{Var}(aX+bY) = a^2 \text{ Var } X + b^2 \text{ Var } Y.$

Proof $E\left[\left(aX+bY - E(aX+bY)\right)^2\right]$

$= E\left[a^2(X-E(X))^2 + b^2(Y-E(Y))^2\right] + 2E[X-E(X)][Y-E(Y)]$

$= a^2 \text{ Var } X + b^2 \text{ Var } Y.$

R Q9 A man with n keys wants to open this door and tries the key ind. and at randoms. Find mean and ~~variable~~ variance of no of trials required to open door.

i) If unsuccessful keys are not eliminate from further selection

ii) if they are.

Sol^n Let man get at first success at x^{th} trial. i.e he is unable to open the door at $(x-1)^{th}$ trial. If unsuccessful keys are not eliminated then X is r.v. which can take 1, 2, 3, - - - -

Prob. of success at first trial $= \frac{1}{n}.$

Prob. of failure at first trial $= 1 - \frac{1}{n}.$

Covariance

$$\text{cov.}(x,y) = E\left[(x-E(x))(y-E(y))\right]$$

Q) A box contain 2^n tickets among which ${}^nC_i$ tickets bear the no. $i = 0, 1, 2, 3, \ldots n$.
A group of m tickets is drawn what is expectation of sum of their numbers.

Sol^n : Let x_i $(i = 1, 2, \ldots m)$ variable respectively no. of i^{th} ticket drawn

Sum of no's on tickets is

$$S = x_1 + x_2 + \cdots + x_m$$

$$S = \sum_{i=1}^{m} x_i$$

$$E(S) = E\left[\sum_{i=1}^{m} x_i\right] = \sum_{i=1}^{m} E(x_i)$$

x_i be r.v. it can take value $1, 2, \ldots n$

$$P(x_i) = \frac{{}^nC_i}{2^n}$$

$$E(x_i) = 1 \cdot \frac{{}^nC_1}{2^n} + 2 \cdot \frac{{}^nC_2}{2^n} + 3 \cdot \frac{{}^nC_3}{2^n} + \cdots$$

$$= \frac{\left[n + \frac{2 \cdot n(n-1)}{2} + \frac{3(n)(n-1)(n-2)}{6} + \cdots\right]}{2^n}$$

$$= \frac{1}{2^n}\left[n + n(n-1) + \frac{n(n-1)(n-2)}{2} + \cdots\right]$$

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n.$$

$$(a+b) = a^n + {}^nC_1 a^{n-1} b +$$

$$\frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2} + \cdots\right]$$

$$= \cancel{\frac{n}{2^n}\left[1+(n-1)\right]^{n-1}}$$

$$= \frac{n}{2^n}\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \cdots + {}^{n-1}C_{n-1}\right]$$

$$= \frac{n}{2^n} \cdot \cancel{2^{n-1}} \cdot (1+1)^{n-1}.$$

$$= \frac{n}{2}.$$

$$E(S) = \sum_{i=1}^{m} \frac{n}{2} = \frac{mn}{2}.$$

Moment Generating f^n (m.g.f)

iy m.g.f of r.v. X is : mgf of random variable X having the prob. $f(x)$

$$M_X(t) = E\left[e^{tX}\right] = \sum e^{tx} p(x) \quad \text{if } X \text{ is discrete}$$

$$= \int e^{tx} f(x) \quad \text{if } X \text{ is cont.}$$

x M.G.F of r.v. X is

x $M_X(t) = E[$

$M_x(t) = E[e^{tx}]$

$= E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots + \frac{t^r x^r}{r!} + \dots\right]$

$= 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \dots +$

$\frac{t^r}{r!}E(x^r) + \dots$

$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r'$

where μ_r' = ~~Expected~~ $E(x^r)$

μ_r' = Moment about origin.

Coefficient of $\frac{t^r}{r!}$ in $\mu_x(t)$ gives μ_r'

∵ $M_x(t)$ generates moment, it is known as m.g.f.

$\mu_r' = \left\{\frac{d^r}{dt^r} M_x(t)\right\}_{t=0}$ ∵ $\frac{d^r}{dt^r}(t^r) = r!$

$\mu_1' = \left(\frac{d}{dt} \cancel{t} M_x(t)\right)$

$\mu_2' = \left(\frac{d^2}{dt^2} M_x(t)\right)$

Theorem: $M_{cx}(t) = M_x(ct)$, c is constant

Proof $M_{cx}(t) = E[e^{tcx}] = E[e^{x(ct)}] = M_x(ct)$

Uniqueness.

Theorem 2: MGF of sum of no. of independent r.v is equal to product of MGF i.e.

$$M_{x_1+x_2+\cdots+x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$$

Proof $M_{x_1+x_2+\cdots x_n}(t) = E\left[e^{t(x_1+x_2+\cdots x_n)}\right]$

$$= E\left[e^{tx_1} \cdot e^{tx_2} \cdot e^{tx_3} \cdots e^{tx_n}\right]$$

$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots E(e^{tx_n}) \quad [\because \text{ind. event}]$$

$$= M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t).$$

H.P.

Theorem 3. Effect of change of origin and scale on MGF.

Let $U = \frac{x-a}{h}$ a and h constant

$$M_U(t) = M_{\frac{x-a}{h}}(t) = E\left[e^{\left(\frac{x-a}{h}\right)t}\right]$$

$$= E\left[e^{\frac{tx}{h} - \frac{at}{h}}\right]$$

$$= e^{-at/h} E\left[e^{tx/h}\right]$$

$$= e^{-at/h} M_x\left(\frac{t}{h}\right)$$

Theorem 4 Uniqueness Theorem of mgf

The m.g.f of distribution exist uniquely determine distribution. This implies the given prob. distribution there is only one mgf and for given m.g.f there is only one distribution.

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \text{----}$

$3ab(a+b)$

Cumulants

$K_x(t) = \log M_x(t)$

$$\Rightarrow K_1 t + \frac{K_2 t^2}{2!} + \cdots + \frac{K_r t^r}{r!} + \cdots = \log M_x(t)$$

$$= \log\left[1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots + \mu_r' \frac{t^r}{r!} + \cdots\right]$$

$$\Rightarrow K_1 t + \frac{K_2 t^2}{2!} + \cdots + \frac{K_r t^r}{r!} = \left(\mu_1' t + \frac{\mu_2' t^2}{2!} + \cdots\right)$$

$$- \frac{1}{2}\left(\mu_1' t + \frac{\mu_2' t^2}{2!} + \cdots\right)^2 + \frac{1}{3}\left(\mu_1' t + \frac{\mu_2' t^2}{2!} + \cdots\right)^3$$

$$+ \cdots$$

$\Rightarrow K_1 = \cancel{\mu_1'}$

$\cancel{K_2/2! = \mu_2'/2! + \mu_1'}$

$\cancel{\frac{K_3}{3!} = \frac{\mu_3'}{3!} + \frac{\mu_1' \mu_2'}{2!}} +$

- $\boxed{K_1 = \mu_1'}$

- $\frac{K_2}{2!} = \frac{\mu_2'}{2!} - \frac{1}{2}\mu_1'^2$

$\boxed{K_2 = \mu_2' - \mu_1'^2} = \mu_2$ Variance

- $\frac{K_3}{3!} = \frac{\mu_3'}{3!} - \frac{3^{\times 2}}{2}\frac{\mu_1' \mu_2'}{\cancel{2}} + \frac{1}{3}\mu_1'^3 - \frac{1}{2}$

$K_3 = \mu_3' - \frac{3}{\cancel{2}}\mu_1' \mu_2' + 2\mu_1'^3.$

$\frac{4\times3\times2}{2\cdot\times2\cdot2}$

$\boxed{K_4}$

$$\times\ \frac{K_4}{4!} = \frac{\mu_4'}{4!} - \frac{1}{2}\cdot\frac{\mu_2'^2}{(2!)^2} - \frac{1}{2}\cdot\frac{2\mu_1'\mu_3'}{3!} + \frac{1}{3}3\left(\frac{\mu_1'^2\mu_2'}{2!} + \frac{\mu_1'^{22}\mu_1'}{(2!)^2}\right)$$

~~$K_4 = \mu_4' - 3\mu_2'^2 - 4\mu_1'\mu_3' + 4\mu_1'^2$~~

$$\frac{K_4}{4!} = \frac{\mu_4'}{4!} - \frac{\mu_2'^2}{2(2!)^2} - \frac{1}{2}\frac{2\cdot\mu_1'\mu_3'}{3!} + \frac{3\mu_1'\mu_2'^2}{3\cdot 2!} - \frac{6\mu_1'^4}{4!}$$

$K_4 = \mu_4' - \mu$

Q $P(X=r) = q^{r-1}p \quad r=1,2,\dots$
Find mgf. of X and hence find mean E(X) and Variance V(X)

Sol^n $M_X(t) = E[e^{tX}] = \sum_{r=1}^{\infty} e^{tr}q^{r-1}p. = \sum_{r=1}^{\infty}\frac{p}{q}(e^tq)^r$

$$= \frac{p}{q}\left[e^tq + (e^tq)^2 + \dots\right]$$

$$= \frac{p}{q}e^tq\left[1 + e^tq + \dots\right]$$

$$= \frac{p}{q}e^tq\left[\frac{1}{1-e^tq}\right]$$

i) $M_x(t) = \frac{pe^t}{1-e^tq}$.

$$\mu_r' = \left[\frac{d^r M_x(t)}{dt^r}\right]_{t=0}$$

Put $r-1$

$$\mu_1' = \left[\frac{d \cdot M_x(t)}{dt}\right]_{t=0}.$$

$$= \left[\frac{d}{dt}\left(\frac{pe^t}{1-e^tq}\right)\right]_{t=0}$$

$$= \frac{(1-e^tq)pe^t - pe^t(-e^tq)}{(1-e^tq)^2} = \left[\frac{pe^t}{(1-e^tq)^2}\right]_{t=0}.$$

$$\mu_1' = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

Put $r=2$.

$$\mu_2' = \left[\frac{d^2 M_x(t)}{dt^2}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{pe^t}{(1-e^tq)^2}\right)\right]_{t=0}.$$

$$= p\left[\frac{(1-e^tq)^2e^t - e^t2(1-e^tq)(-e^tq)}{(1-e^tq)^4}\right]$$

$$= p\left[\frac{e^t(1-e^tq)+2e^{2t}q}{(1-e^tq)^3}\right]$$

$$= p\left[\frac{e^t + e^{2t}q}{(1-e^tq)^3}\right]_{t=0} = p\left[\frac{1+q}{(1-q)^3}\right] = p\left[\frac{2-p}{p^3}\right].$$

$$\mu_2' = \frac{2-p}{p^2} = \frac{1+q}{p^2}.$$

Variance $= \mu_2' - (\mu_1')^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

Q Find MGF of Random Variable whose moments are

$$\mu_r' = (r+1)!\,2^r$$

$$M_x(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \text{---} + \frac{\mu_r' t^r}{r!} + \text{---}.$$

$$= \sum_{r=0}^{\infty} \frac{\mu_r' t^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{(r+1)!\,2^r t^r}{r!} = \sum_{r=0}^{\infty} (r+1)\,2^r t^r.$$

$$= \sum_{r=0}^{\infty} 1 + 2\cdot 2t + 3\cdot 2^2 t^2 + 4\cdot 2^3 t^3 + \cdots$$

$$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \text{--}$$

$$= [1-2t]^{-2}.$$

Q If $\mu_r' = r!$ Find mgf of X.

$$M_X(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + ---$$

$$= \sum_{r=0}^{\infty} \frac{\mu_r' t^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{r!\, t^r}{r!}$$

$$= 1 + t + t^2 + --$$

$$= (1-t)^{-1}.$$

Q A r.v of prob. f^n $p(x) = \frac{1}{2^x}$ $x = 1, 2, 3, \ldots$

find mgf, mean, variance of X

Sol^n ~~$M_X(t) = \sum_{r=0}^{\infty}$~~

$$M_X(t) = E[e^{tx}] = \sum_{r=1}^{\infty} e^{tr} \frac{1}{2^r} = \frac{e^t}{2} + \frac{e^{2t}}{2^2} +$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \frac{e^{2t}}{2^2} + -\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2} \times \frac{2}{2-e^t}$$

$$= \frac{e^t}{2-e^t}.$$

$E(x^2) - [E(x)]^2 = ~~12 - 4~~ \; 6 - 4$
$= 2 .$

$$\mu_r' = \frac{d^r}{dx^r}\left[\frac{e^t}{2-e^t}\right]_{t=0} \Rightarrow$$

$$\mu_1' = \frac{d}{dt}\left[\frac{e^t}{(2-e^t)}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)\,e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \frac{2e^t}{(2-e^t)^2} = 2 .$$

$$\mu_2' = \frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right] = \frac{(2-e^t)^2\,2e^t - 2e^t(2-e^t)}{(2-e^t)^4}$$

$$= \frac{(2-1)^2\,2+4}{} = 6$$

$Var = 6 - 4 = 2 .$

Imp Q If μ_r' is r^{th} moment about origin. Prove that

$$\mu_r' = \sum_{j=1}^{r} {}^{r-1}C_{j-1}\, \mu'_{r-j}\, k_j$$

Sol^n $K_x(t) = \log M_x(t)$.

$$k_1 t + \frac{k_2 t^2}{2!} + \cdots + \frac{k_{r-1} t^{r-1}}{(r-1)!} + \frac{k_r t^r}{r!} = \log\left[1 + \mu_1' t + \frac{\mu_2' t^2}{2!} + \cdots + \frac{\mu_r' t^r}{r!}\right]$$

diff. wrt. t

$$k_1 + k_2 t + \cdots + \frac{k_3 t^2}{2!} + \cdots + \frac{k_{r-1} t^{r-2}}{(r-2)!} + \frac{k_r t^{r-1}}{(r-1)!} + \cdots =$$

$$\frac{\mu_1' + \mu_2' t + \mu_3' t^2/2! + \cdots + \frac{\mu_r' t^{r-1}}{(r-1)!}}{1 + \mu_1' t + \frac{\mu_2' t^2}{2!} + \cdots + \frac{\mu_r' t^r}{r!}}$$

$$\Rightarrow \left(k_1 + k_2 t + \frac{k_3 t^2}{2!} + \cdots + \frac{k_r t^{r-1}}{(r-1)!} + \cdots\right)\left(1 + \mu_1' t + \frac{\mu_2' t^2}{2!} + \cdots + \frac{\mu_r'}{r!}\right)$$

$$= \mu_1' + \mu_2' t + \cdots + \frac{\mu_r' t^{r-1}}{(r-1)!} + \cdots$$

~~K~~ Collecting coefficient of $\frac{t^{r-1}}{(r-1)!}$ to both side

$$\mu_r' = \frac{k_r}{(r-1)!} + \frac{\mu_{r-1}' k_1}{(r-1)!} + \frac{\mu_{r-2}' k_2}{(r-2)!} + \cdots$$

$$= \sum_{j=1}^{r} k_j \mu_{r-j}'$$

$\frac{(r-1)!}{(j-1)!\,(r-j)!}$

$\Rightarrow k_1 \mu'_{r-1} + (r-1) k_2 \mu'_{r-2} + \dots + k_r t^{r-1}.$

$\Rightarrow {}^{r-1}C_0 k_1 \mu'_{r-1} + {}^{r-1}C_1 k_2 \mu'_{r-2} + \dots + {}^{r-1}C_{r-1} \mu'_0 k_r = \mu'_r$

$\Rightarrow \mu'_r = \sum_{j=1}^{r} {}^{r-1}C_{j-1}\, \mu'_{r-j}\, k_j$

Characteristic function

$$\phi_x(t) = E[e^{itx}] = \begin{cases} \int e^{itx} p(x)\,dx \longrightarrow \text{Continous} \\ \sum e^{itx} p(x) \longrightarrow \text{Discrete} \end{cases}$$

Properties

i) $\phi(0) = 1.$

ii) $|\phi(t)| \leq 1$

iii) $\phi(t)$ is continous everywhere.

iv) $\underset{t\to\infty}{Lt}\ \phi(t) = 0$

Relation with μ'_r

$$\boxed{\mu'_r = (-1)^r \left[\frac{\partial^r}{\partial t^r}\phi(t)\right]_{t=0}}$$

Relation with Cumulants

$$K(t) = \sum_{r=1}^{\infty} \frac{(it)^r k_r}{r!}$$

$K(t) \longrightarrow \log M_x(t)$

$K(t) \longrightarrow \log$ ~~$M_x(t)$~~ $\phi_x(t)$ character f^n

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

Obtain Characteristic f^n of Gamma. dist^n.

$k_r = n(r-1)!$

Sol^n $k(t) = \sum_{r=1}^{\infty} \frac{(it)^r k_r}{r!}$

~~k(t)~~ $\log\phi(t) = \sum_{r=1}^{\infty} \frac{(it)^r \cdot n(r-1)!}{r!}$

$= n \sum_{r=1}^{\infty} \frac{(it)^r}{r}$

$= n.\left[\frac{it}{1} + \frac{(it)^2}{2} + \frac{(it)^3}{3} \cdots\right]$

$\log\phi(t) = \log[1-it]^{-n}$

$\phi_x(t) = (1-it)^{-n}$

Imp **Inversion Theorem.**

If $\phi(t)$ is characteristic function and $F(x)$ is distribution function then density function $f(x)$ is given by

pdf $f(x) = F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$

$f(x) = \frac{dF}{dx}$

pdf cdf/df

$F(x) = P(X \le x)$

$F(x) = \int_{-\infty}^{x} f(x)\,dx$

$-\log(1-x)$ $\qquad$ $|t| = -t \quad t<0$

$\qquad\qquad\qquad\quad t \quad t>0$

PAGE NO.:
DATE: / /

Q Show that distribution for which characteristic f^n is $e^{-|t|}$ has density f^n $f(x) = \frac{1}{\pi(1+x^2)}$ Cauchy's distribution.

Sol^n From Inversion Theorem.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} e^{-|t|} dt.$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-itx} e^{-|t|} dt + \int_{0}^{\infty} e^{-itx} e^{-|t|} dt\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-itx} e^{t} dt + \int_{0}^{\infty} e^{-itx} e^{-t} dt\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-t(ix-1)} dt + \int_{0}^{\infty} e^{-t(ix+1)} dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{t(-ix+1)}}{+(-ix+1)}\right]_{-\infty}^{0} + \left[\frac{e^{-t(ix+1)}}{-(ix+1)}\right]_{0}^{\infty}.$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-ix} + \frac{1}{1+ix}\right]$$

$$= \frac{1}{2\pi} \times \frac{2}{1+x^2} = \frac{1}{\pi(1+x^2)}$$

Chebychev's Inequality

If X is r.v. with mean μ and variance σ^2 then for any positive no. k, we have

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{K^2}$$

or $$P\{|X-\mu| < k\sigma\} \geq 1-\frac{1}{K^2}.$$

In particular if we take $k\sigma = c$.

$$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$$

$$P\{|X-\mu| < c\} \geq 1-\frac{\sigma^2}{c^2}$$

$$= 1-\frac{\text{Var}\,X}{c^2}.$$

Q If $p(x) = 2^{-x}$, $x = 1, 2, \ldots$

Prove that Chebychev's Inequality

$P\{|X-2| \leq 2\} > \frac{1}{2}$. Also find actual prob.

Solⁿ $E(X) = \cancel{2}.\ \sum x\,p(x) = \sum_{x=1}^{\infty} x.2^{-x}.$

μ_1' / μ (Mean)

$= 2.$

$\Rightarrow E(X) = 2.$

$$E(X^2) = \sum x^2 p(x) = \sum \frac{x^2}{2^x} = \frac{1}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \cdots$$

$= \frac{1}{2}[1 + 1$

$-(x-2)$

σ = $\text{Var } X = E(x^2) - [E(x)]^2$

$= 6-4$

$= 2$.

from chebychev's inequality

$P\{|x-\mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$

$P\{|x-2| \leq \sqrt{2}k\} > 1 - \frac{1}{k^2}$.

Put $k = \sqrt{2}$.

$P\{|x-2| \leq 2\} > 1 - \frac{1}{(\sqrt{2})^2}$.

$P\{|X-2| \leq 2\} \geq \frac{1}{2}$.

Actual Prob.

$P\{|x-2| \leq 2\}$.

$\rightarrow P\{0 \leq x \leq 4\} = P\{x=1\} + P\{x=2\} + P\{x=3\} + P\{x=4\}$

$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

$= \frac{8+4+2+1}{2^4} = \frac{15}{16}$

Q If X is no. scored in throw of fair dice show Chebychev's ineq. give

$P\{|X-\mu| > 2.5\} \le 0.4$ actual prob is zer.

Solⁿ

1

x	1	2	3	4	5	6
p(x)	1/6	1/6	1/6	1/6	1/6	1/6

$E(x) = \sum x\, p(x) = \frac{7}{2}.$

$E(x^2) = \frac{1}{6}[1+4+9+16+25+36]$

$= \frac{1}{6}[55] = \frac{55+36}{6} = \frac{91}{6}$

$\sigma^2 = \frac{91}{6} - \frac{49}{4}$

$\sigma^2 = \frac{182 - 147}{12} = \frac{35}{12} = 2.81.$

$$\begin{array}{r} 02.8 \\ 12\overline{)35} \\ 24 \\ \hline 110 \\ 108 \\ \hline 20 \end{array}$$

$P\left\{\left|X - \frac{7}{2}\right| > c\right\} < \frac{\sigma^2}{c^2}.$

Take $c = 2.5$

35

$P\left\{\left|X - \frac{7}{2}\right| > 2.5\right\} < \frac{35}{12\cdot(6.25)}$

$P\left\{\left|X - \frac{7}{2}\right| > 2.5\right\} < \frac{35}{75}$

$P\left\{\left|X - \frac{7}{2}\right| > 2.5\right\} < 0.47$

$$\begin{array}{r} 6.25 \\ 12 \\ \hline 1250 \\ 625 \\ \hline 75.00 \end{array}$$

$$\begin{array}{r} 0.41 \\ 7.5\overline{)3.50} \\ 390 \\ \hline 100 \end{array}$$

PAGE NO.:
DATE: / /

~~$P\{1 \le x \le 6\}$~~

$P\{x < 1\} + P\{x > 6\}$
$0 + 0$
$= 0$.

Q Two unbiased dice are thrown. If X is sum of no's showing up. Prove

NR

$$P\{|X - 7| \ge 3\} \le \frac{35}{54}.$$

Sol^n

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$E(X) = \Sigma x \cdot f(x)$
$= \frac{252}{36} = 7.$

$$E(X^2) = \frac{4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144}{36}$$

$= \frac{1}{36}(1974) = \frac{329}{6}.$

$Var(X) = E(X^2) - [E(X)]^2$
$= \frac{329}{6} - 49$
$= \cancel{329 -} \frac{35}{6}$

$x-7>3$ $\quad$ $(x-7)<-3$
$x>10$ $\quad$ $x<4$

PAGE NO.:
DATE: / /

By Chebychev's Inequality

$$P\{|X-\mu|\geq c\} \leq \frac{\operatorname{Var} x}{c^2}$$

Take $c = 3$

$$P\{|X-7\geq\}\cancel{\varepsilon} \leq \frac{35}{9\times 6}$$

$$P\{|X-7|>3\}\cancel{\varepsilon} \leq \frac{35}{54} \quad \text{H.P}$$

Actual Prob.

$P(x<4) + P(x\geq10)$

$\frac{12}{36} = \frac{1}{3}$

Q. A symmetric die is tossed 720 times. Use Chebychev's to find lower bound of getting 100 to 140 sixes.

NR

Solⁿ

p = Probability of success in a single throw
$= 1/6$

$q - 5/6$.

$$\text{Mean} = \mu = np = 720 \times \frac{1}{6}$$
$$= 120.$$

$$\text{Variance } \sigma^2 = npq = 720\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$$
$$= 100.$$

To find lower bound of
$P(100 < x < 140)$

$$P\{|X-\mu| < k\sigma\} \geq 1-\frac{1}{k^2}.$$

Scanned with CamScanner

$P\{-k\sigma < X-\mu < k\sigma\} \geq 1-1/k^2$

$P\{\mu-k\sigma < X < \mu+k\sigma\} \geq 1-1/k^2$

$\mu - k\sigma = 100$ $\qquad$ $120-10k=100$

$\mu + k\sigma = 140$ $\qquad$ $120+10k=140$

$2\mu = 240$ $\qquad$ $\boxed{k=2}$

$\boxed{\mu=120}$

$P\{100 < x < 140\} \geq 1-\frac{1}{2^2}.$

$P\{100 < x < 140\} \geq 3/4.$

Least bound : 0.75

Q The exp. is to ~~to~~ toss 2 balls in 5 boxes in such a way that each ball is equally likely to fall in any box. Let X denote the no. of balls in first box.

i) find pdf.

Let B_i $i=1,2$ be the events the ball falls into 1^{st} box

$P(B_1) = 1/5 = P(B_2)$

$P(X=0) = P(B_1^c \cap B_2^c)$

$= (1-1/5)(1-1/5) = \frac{16}{25}$

$P(X=1) = P[(B_1^c \cap B_2) \cup (B_1 \cap B_2^c)]$

$= (1-1/5)\frac{1}{5} + \frac{1}{5}(1-1/5) = \frac{8}{25}$

$P(X=2) = 1/25$

$$f(x) = \begin{cases} 16/25 & x=0 \\ 8/25 & x=1 \\ 1/25 & x=2, -1 \\ 0 & \text{Otherwise.} \end{cases}$$

Q A random variable x has d.f e^{-x} $x \geq 0$

NP Show Chebychev's inequality gives $P(|x-1|>2) < 1/4$ and show actual prob is e^{-3}

Sol: $f(x) = e^{-x}$ $x \geq 0$.

x follows exponential distribution $\lambda = 1$

$[\because$ for " " $f(x) = \lambda e^{-\lambda x}]$

$\mu = E(x) = 1/\lambda$ $\mu = \frac{1}{1} = 1$

$\sigma^2 = \frac{1}{\lambda^2} = 1$

$\boxed{\sigma = 1}$ $\boxed{\mu = 1}$

Chebychev's

$$P\{|x-\mu| > k\sigma\} < \frac{1}{k^2}$$

$$P\{|x-1| > 2\}$$

$\mu = 1$ $k = 2$.

$$P\{|x-1| > 2\} < 1/4.$$

PAGE NO. :
DATE : / /

H.P.

$$P\{|x-1| > 2\} = 1 - P\{|x-1| < 2\}$$
$$= 1 - P\{-1 < x < 3\}$$
$$= 1 - P\{0 < x < 3\}$$
$$= 1 - \int_0^3 f(x)\,dx$$
$$= 1 - \int_0^3 e^{-x}\,dx.$$
$$= 1 + [e^{-x}]_0^3$$
$$= 1 + [e^{-3} - e^0]$$
$$= e^{-3}.$$

Q An unbiased coin is tossed 100 times. Show

N^n $P\{30 < x < 70\} \geq 0.93$

Sol^n

$$\mu = 100\left(\frac{1}{2}\right) = 50$$

$$\sigma^2 = 100\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 25$$

$$\boxed{\sigma = 5}$$

$$P\{|x - \mu| < k\sigma\} > 1 - \frac{1}{k^2}$$

$$\mu - k\sigma = 30$$
$$\mu + k\sigma = 70$$
$$2k\sigma = 40$$
$$\boxed{k = 4}$$

$$P\{|x - 50| < 20\} > 6\ 1 - \frac{1}{16} \approx 0.93.$$

Q. Use Chebyshev's inequality to determine how many times a fair coin must tossed in order that the probability will be at least 0.95 that the ratio of the no. of head to no. of tosses will be b/w 0.45 & 0.55

Solⁿ Let X be the no. of heads obtained when a fair coin is tossed n times.

$p = 1/2 \quad q = 1/2.$

Mean = np = $\quad Var(x) = npq$

Mean of required ratio $\frac{X}{n} := E\left(\frac{1}{n}X\right)$

$= \frac{1}{n}E(X) = \frac{1}{n}.np = 1/2$

$\boxed{\mu = 1/2}$

$Var\left(\frac{X}{n}\right) = \left(\frac{1}{n}\right)^2 Var(x)$

$= \frac{1}{n^2}\cdot npq = \frac{pq}{n} = \frac{1}{4n}.$

$\boxed{\sigma = \frac{1}{2\sqrt{n}}}.$

Using Chebyshev's In.

$P\left\{\left|\frac{X}{n} - \mu\right| \leq k\sigma\right\} \geq 1 - \frac{1}{k^2}$

$\mu - k\sigma = 0.45$

$\mu + k\sigma = 0.55$

$2k\sigma = 0.10.$

$k\sigma = 0.05$

$\frac{k}{2\sqrt{n}} = 0.05.$?

$\frac{k}{\sqrt{n}} = 0.10$

$\frac{1}{k} = \frac{\sqrt{n}}{0.10}$

$1 - 1/k^2 = 0.95$ $1/k^2 = 0.05$

$k^2 = 20$

~~$1 - \frac{n}{0.01} = 0.95$~~

~~$1 - 0.95 = \frac{n \times 100}{.0001}$~~

~~$0.05 = n \times 100.$~~

$\sqrt{n} = \frac{k}{0.10}$

$n = \frac{k^2}{0.01} = 2000$

Imp

Q A symmetric die is thrown 600 times. find lower bound for probability of getting 80 to 120 sixe

NR

Solⁿ Let S be total no. of success.

$E(S) = np = 600\left(\frac{1}{6}\right) = 100$

$V(S) = npq = \cancel{600}^{\overset{50}{100}}\left(\frac{1}{\cancel{6}}\right)\left(\frac{5}{\cancel{6}_3}\right) = \frac{250}{3} - \frac{500}{6}.$

Using this Chebchev's ineq

$P\left\{|S - E(S)| < k\sigma\right\} \geq 1 - \frac{1}{k^2}$

$P\left\{|S - 100| < k\sqrt{\frac{500}{6}}\right\} \geq 1 - \frac{1}{k^2}.$

$P\left\{100 - k\sqrt{\frac{500}{6}} < S < 100 + k\sqrt{\frac{500}{6}}\right\}$

$$\therefore 80 = \left\{100 - k \cdot \sqrt{\frac{500}{6}}\right.$$

$$120 = 100 + k\sqrt{\frac{500}{6}}$$

$$40 = 2k\sqrt{\frac{500}{6}}$$

$$\frac{\overset{20}{\cancel{40}}}{\cancel{2}}\sqrt{\frac{6}{500}} = k$$

$$20 \cdot 26 \sqrt{\frac{6 \cdot 400}{500}} \geq k$$

Take $k = \frac{20}{\sqrt{500/6}}$

$$P\{80 < S < 120\} \geq 1 - \frac{500}{400 \cdot 6}$$

$$P\{80 < S < 120\} \geq 1 - \frac{5}{24}$$

$$P\{80 < S < 120\} \geq \frac{19}{24}$$

only statement

Central Limit Theorem

Let $X_1, X_2, --- X_n$ be n-independent random variable which have same distribution

Let common expectation μ and Variance σ^2

Let $\bar{x} = \frac{1}{n}\sum X_i$, then distribution of $\bar{x}$ approaches Normal distribution with mean μ Variance σ^2/n.

$$E(\bar{y}) = E\left(\frac{1}{n}\Sigma X_i\right) = \frac{1}{n}E(\Sigma x_i)$$

$$= \frac{1}{n}\Sigma E(x_i)$$

$$= \frac{\Sigma \mu}{n} = \frac{n\mu}{n} = \mu .$$

~~$V(x_i) = \sigma^2$~~ .

$$V(\bar{x}) = Var\left(\frac{\Sigma x_i}{n}\right) = \frac{1}{n^2} Var(\Sigma x_i)$$

$$= \frac{1}{n^2} Var(X_1 + X_2 + \cdots + X_n)$$

$$= \frac{1}{n^2}\left[Var(x_1) + Var(x_2) + \cdots + Var(x_n)\right]$$

$$= \frac{1}{n^2} \cdot \underbrace{\left[\sigma^2 + \sigma^2 + \cdots + \sigma^2\right]}_{n \text{ times}}$$

$$= \frac{\sigma^2}{n} .$$