हमेशा dist^n funct^n देकर [illegible] ch. funct^n ch. funct^n दिया है तो dist^n निकालना है

Unit – II

V. Imp.

Inversion theorem on characteristic function (Or Laplace transforms)

Result → (i) $\left|\frac{\sin\theta}{\theta}\right| < 1$ ——— (A)

(ii) $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right) dx = \pi$

Definition → Continuity interval : An interval $(a-h, a+h)$ is said to be a continuity interval of a dist^n function $F(x)$ if $F(x)$ is continuous at end points $\{a-h$ and $a+h\}$ of the given interval

Statement :→ If $(a-h, a+h)$ is continuity interval of dist^n funct^n $f_x(x)$ of or continuous r.v. X, then

$$f(a+h) - f(a-h) = \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{+T} e^{-iat} \frac{\sin ht}{t} \phi(t)\, dt$$

where is $\phi(t)$ = ch. funct^n of a r.v. X.

(ii) $a \in R(-\infty, \infty)$

(iii) $h \to 0$ i.e. $|h| < \infty$

Proof :→ R.H.S. $= \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} e^{-iat} \frac{\sin ht}{t}$

$\because \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$ $\{ \int_{-\infty}^{\infty} e^{itx} dF(x)$

$dF(x)$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{\pi} \lim_{T\to 0} \int_{-T}^{T} \left\{ \int_{-\infty}^{\infty} e^{it(x-a)} \frac{\sin ht}{t} \, dF(x) \right\} dt \qquad \text{—— (1)}$$

and $\left| e^{it(x-a)} \cdot \frac{\sin ht}{t} \right| \leq \left| e^{it(x-a)} \right| \left| \frac{\sin ht}{ht} \right| |h|$ ↳ from (A)

$\leq 1 \cdot 1 \cdot |h|$

$\leq |h| < \infty$ as $h \to 0$

hence interchanging the order of integration in (1) by Fubini's thm. we get;

$$\text{R.H.S.} = \frac{1}{\pi} \lim_{T\to\infty} \int_{-\infty}^{\infty} \left\{ \int_{-T}^{T} e^{it(x-a)} \frac{\sin ht}{t} dt \right\} dF(x)$$

$$= \frac{1}{\pi} \lim_{T\to\infty} \int_{-\infty}^{\infty} G(T,x) \, dF(x) \qquad \text{—— (2)}$$

where $G(T,x) = \int_{-T}^{T} e^{it(x-a)} \frac{\sin ht}{t} dt$

$$= \int_{-T}^{T} \{ \cos t(x-a) + i \sin t(x-a) \} \frac{\sin ht}{t} dt$$

$$= \int_{-T}^{T} \cos t(x-a) \frac{\sin ht}{t} dt + i \underbrace{\int_{-T}^{T} \sin t(x-a) \frac{\sin ht}{t} dt}_{=0}$$

$2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$= \frac{1}{2}\int_{-T}^{T} 2\cos t(x-a) \frac{\sin ht}{t}\, dt$$

$$= \frac{1}{2}\int_{-T}^{T} \frac{1}{t}\{\sin(x-a+h)t - \sin(x-a-h)t\}\, dt$$

$$G(T,x) = \frac{1}{2}\int_{-T}^{T} \frac{\sin qt}{qt}\, q\, dt - \frac{1}{2}\int_{-T}^{T} \frac{\sin pt}{pt}\, p\, dt \quad \text{--- (3)}$$

where (i) $q = x-a+h = x-(a-h)$
(ii) $p = x-a-h = x-(a+h)$ --- (2.1)

Since $\left|\frac{\sin\theta}{\theta}\right| < 1$ hence $G(T,x) < \infty$

i.e. bounded and hence taking $\lim_{T\to\infty}$ inside the integrat^n in (2)

we have

$$R.H.S. = \frac{1}{\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} G(T,x)\, dF(x) \quad \text{--- (4)}$$

$$= \lim_{T\to\infty} \frac{1}{\pi}\left[\int_{-\infty}^{a-h-\epsilon} G(T,x)\, dF(x) + \int_{a-h-\epsilon}^{a-h} ''\right.$$

limit has breaked into parts

$$+\int_{a-h}^{a+h-\epsilon} ,, + \int_{a+h-\epsilon}^{a+h} ,, + \int_{a+h}^{\infty} ,, \Big] \quad \text{--- (5)}$$

in eq^n (3) put $qt = y$ & $qdt = dy$ and $pt = z$, $pdt = dz$

then eq^n (3) becomes;

$$G(T, x) = \frac{1}{2}\int_{-qT}^{qT} \frac{\sin y}{y}\,dy - \frac{1}{2}\int_{-pT}^{pT} \frac{\sin z}{z}\,dz \quad \text{--- (3.1)}$$

eqn.

<u>Case-I</u> $x < a-h$ then $q < 0$ and $p < 0$ by (2.1) and (3.1)

$$\lim_{T\to\infty} G(T, x) = \frac{1}{2}\int_{\infty}^{-\infty} \frac{\sin y}{y}\,dy - \frac{1}{2}\int_{\infty}^{-\infty} \frac{\sin z}{z}\,dz$$

$$= -\frac{\pi}{2} + \frac{\pi}{2} = 0$$

[$\because q<0$ limit में $q = -ve$ होगा तो $\int_{+qT}$ (+ve हो जायेगी) $\because T \to \infty$ $\int_{\infty}^{-\infty}$

<u>Case-II</u> $x = a-h$ then $q = 0$ and $p < 0$ by (2.1)

$$\lim_{T\to\infty} G(T, x) = \frac{1}{2}\int_{0}^{0} \frac{\sin y}{y}\,dy - \frac{1}{2}\int_{\infty}^{-\infty} \frac{\sin z}{z}\,dz$$

$$= 0 + \pi/2 = \pi/2$$

<u>Case-III</u> $a-h < x < a+h$ then $q > 0$ and $p < 0$

$$\lim_{T\to\infty} G(T, x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy - \frac{1}{2}\int_{\infty}^{-\infty} \frac{\sin z}{z}\,dz$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

<u>Case-IV</u> $x = a+h$ then $q > 0$ and $p = 0$

by (2.1)

$$\lim_{T\to\infty} G(T,x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy - 0 = \frac{\pi}{2}$$

<u>Case-V</u>

$x > a+h$ then $p>0$ & $q>0$ by (2.1)

$$\lim_{T\to\infty} G(T,x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin z}{z}\,dz$$

$$= \frac{\pi}{2} - \frac{\pi}{2} = 0$$

Thus
$$\lim_{T\to\infty} G(T,x) = \begin{cases} 0 & \text{for } x<a-h \\ \pi/2 & \text{for } x=a-h \\ \pi & \text{for } a-h<x<a+h \\ \pi/2 & \text{for } x=a+h \\ 0 & \text{for } x>a+h \end{cases} \qquad (6)$$

Substitute (6) in eq^n (5)

R.H.S.
$$= \lim_{\epsilon\to 0} \frac{1}{\pi}\left[\int_{-\infty}^{a-h-\epsilon} 0\cdot dF(x) + \int_{a-h-\epsilon}^{a-h} \frac{\pi}{2}\,dF(x) + \int_{a-h}^{a+h-\epsilon} \pi\,dF(x) + \int_{a+h-\epsilon}^{a+h} \frac{\pi}{2}\,dF(x) + \int_{a+h}^{\infty} 0\cdot dF(x)\right]$$

$$= \frac{1}{\pi}\lim_{\epsilon\to 0}\Big[0 + \frac{\pi}{2}\{F(a-h) - F(a-h-\epsilon)\}$$
$$+ \pi\{F(a+h-\epsilon) - F(a-h)\} + \frac{\pi}{2}\{F(a+h) - F(a+h-\epsilon)\}$$

$$\lim_{\varepsilon \to 0} \left[\frac{1}{2}\{f(a+h)+f(a+h-\varepsilon)\} - \frac{1}{2}\{f(a-h)+f(a-h-\varepsilon)\}\right]$$

$$= \underbrace{\frac{1}{2}\{f(a+h)+f(a+h)\} - \frac{1}{2}\{f(a-h)+f(a-h)\}}_{=\ f(a+h)-f(a-h)}$$

$\left[\because \lim_{\varepsilon \to 0} f(a+h-\varepsilon) = f(a+h)\right.$ and

$\lim_{\varepsilon \to 0} f(a-h-\varepsilon) = f(a-h)$ for $(a-h, a+h)$

being continuity interval of $f(x)$ $\left.\right]$

$f(a+h) - f(a-h) = L.H.S.$ H.P.

Corollary (i) :→ If $\phi(t)$ is integrable absolutely then p.d.f. $f(x)$ of a continuous r.v. X is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi(t)\,dt$$

Proof :→ we have

$$f(a) = \left[\frac{d}{dx} f(x)\right]_{x=a}$$

$$\left[\lim_{h\to 0}\left\{\frac{f(a+h)-f(a-h)}{2h}\right\}\right] \quad \text{— (1)}$$

by mean value theorem of diff. Calculus]

Substitute the result of $f(a+h)-f(a-h)$ by inversion thm.

$$\frac{1}{2\pi}\lim_{\substack{h\to 0\\ T\to\infty}}\int_{-T}^{T} e^{-iat}\,\frac{\sin ht}{ht}\,\phi(t)\,dt \quad \text{— (2)}$$

But $\left| e^{-iat}\,\frac{\sin ht}{ht}\,\phi(t)\right| < \left|e^{-iat}\right|\left|\frac{\sin ht}{ht}\right|\left|\phi(t)\right|$

$$< |\phi(t)|$$

i.e. it is integrable and hence taking limit $h\to 0$ inside the integral sign in eq^n. (2) we have

$$f(a) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \lim_{h\to 0} e^{-iat} \left(\frac{\sin ht}{ht}\right) \phi(t)\,dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iat} (1)\, \phi(t)\,dt$$

Replace a by x

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\, \phi(t)\,dt$$

Uniqueness Theorem :→ If two dist^n funct^n $f_1(x)$ and $f_2(x)$ have the same ch. function $\phi_1(t)$ and $\phi_2(t)$ then $f_1(x) = f_2(x)$

Proof :→ Let (a, b) be the ~~Conti.~~ continuous point of the dist^n function $F_1(x)$ and $F_2(x)$

then $F_1(b) - F_1(a) = F_2(b) - F_2(a)$

$$= \frac{1}{\pi} \lim_{T\to\infty} \int_{-T}^{T} e^{-iat} \frac{\sin ht}{ht} \phi(t)\,dt \quad \text{———} \ (1)$$

where $\phi_1(t) = \phi_2(t) = \phi(t)$

taking $\lim_{a\to-\infty}$ we have $F_1(-\infty) = 0 = F_2(-\infty)$

$f_1(b) - 0 = f_2(b) - 0$

or $F_1(b) = F_2(b)$

replace b by x

$F_1(x) = F_2(x)$

Corollary (iii) :→ A necessary and sufficient condition for two distⁿ function with p.d.f.

$f_1(x)$ and $f_2(x)$ to be identical is that the ch. funcⁿ $\phi_1(t)$ and $\phi_2(t)$ are identical.

Proof :- Necessary condition

given $f_1(x) = f_2(x)$ ——— ①

to prove $\phi_1(t) = \phi_2(t)$

we have by defⁿ

$$\phi_1(t) = E[e^{itx}]$$

$$= \int_{-\infty}^{\infty} e^{itx} f_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{itx} f_2(x)\,dx$$

$$= \phi_2(t) \quad \text{from ②}$$

Sufficient Condition :→

Given $\phi_1(t) = \phi_2(t)$
to prove $f_1(x) = f_2(x)$
by corr. (1) and inversion thm.

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi_1(t)\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi_2(t)\,dt$$

$$= f_2(x)$$ Hence proved

Q.1 Obtain the p.d.f. of r.v. X for which ch. funct^{n}s are

(i) $\phi(t) = e^{-t^2/2}$

Sol.^{n} :→ $\boxed{\text{Result (1)}}$ $\displaystyle\int_{-\infty}^{\infty} \frac{e^{\pm itx}}{(\alpha \pm it)^p}\,dt = \frac{2\pi}{\Gamma p} e^{-\alpha x} x^{p-1}$

$\boxed{2}$ $\displaystyle\int_{-\infty}^{\infty} \frac{e^{\pm itx}}{(1+t^2)}\,dt = \pi e^{-|x|}$

(1) By inversion theorem
from corollary(i) $\displaystyle f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi(t)\,dt$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$ $\qquad z = \frac{x-\mu}{\sigma}$ $\qquad t^2 + 2itx + i^2x^2$

$dz = \frac{dx}{\sigma}$ $\qquad \boxed{dx = \sigma\, dz}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} \cdot e^{-t^2/2}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\{t^2 + 2itx - x^2 + x^2\}}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\{(t+ix)^2 + x^2\}}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{-\frac{1}{2}(t+ix)^2}\, dt$$

$$= \frac{1}{2\pi}\, e^{-x^2/2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2}\, dt$$

[Normal distn $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
$N \sim (-ix, 1)$ mean & var.]

$$= \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \times 1 \qquad \because \int_{-\infty}^{\infty} f(x)\,dx = 1$$

[as $t \sim N(-ix, 1)$]

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \longrightarrow \text{S.N.V.} \sim N(0,1)$$