(ii) $\phi(t) = e^{-|t|}$

By inversion theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, e^{-|t|}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx-|t|}\,dt$$

Note: $\{-|t|\}$ It should be -ve after removing modulus

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-itx+t}\,dt + \int_{0}^{\infty} e^{-itx-t}\,dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{t(1-ix)}}{(1-ix)}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[-\frac{e^{-t(ix+1)}}{(ix+1)}\right]_{0}^{\infty}$$

$$\frac{1}{2\pi(1-ix)}\left[e^{0} - e^{-\infty}\right] + \frac{1}{2\pi(ix+1)}\left[-e^{-\infty} + e^{0}\right]$$

$$\frac{1}{2\pi(1-ix)}\left[1 - e^{-\infty}\right] + \frac{1}{2\pi(ix+1)}\left[-e^{-\infty} + 1\right]$$

$$= \frac{1}{2\pi(1-ix)}[1-0] + \frac{1}{2\pi(1+ix)}[0+1]$$

$$= \frac{1}{2\pi}\left[\cancel{i+ix+} \frac{1}{(1-ix)} + \frac{1}{1+ix} \right]$$

$$= \frac{1}{2\pi}\left[\frac{1+\cancel{ix}+1-\cancel{ix}}{\cancel{i}(1-ix)(1+ix)} \right]$$

$$= \frac{1}{\cancel{2}\pi} \cdot \frac{\cancel{2}}{1+x^2} = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

Cauchy distⁿ

(iii) $\phi(t) = \dfrac{1}{(1-2it)^{n/2}}$ even

Solution:→ $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-itx} \frac{1}{(1-2it)^{n/2}}\, dt$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}}{2^{n/2}\left(\frac{1}{2}-it\right)^{n/2}}\, dt$$

[by result ①

$\alpha = \frac{1}{2}$ $p = \frac{n}{2}$

$$= \frac{1}{2\pi} \cdot \frac{2\pi}{2^{n/2}\,\Gamma\frac{n}{2}}\, e^{-x/2}\, x^{n/2-1}$$

$$f(x) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}}\, e^{-x/2}\, x^{n/2-1}$$ Prob. fᵘⁿᶜᵗ of chi-square distⁿ

is $\phi(t) = e^{i\mu t - \frac{t^2\sigma^2}{2}}$

Ans:- By inversion theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} e^{i\mu t - \frac{t^2\sigma^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx + i\mu t - t^2\sigma^2/2}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it(x-\mu) - \frac{t^2\sigma^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\{t^2\sigma^2 + 2(x-\mu)it\}}\,dt \quad i^2 = -1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t^2\sigma^2 + 2(x-\mu)it + i^2\frac{(x-\mu)^2}{\sigma^2} - i^2\frac{(x-\mu)^2}{\sigma^2}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(t^2\sigma^2 + 2(x-\mu)it - \frac{(x-\mu)^2}{\sigma^2} + \frac{(x-\mu)^2}{\sigma^2}\right)}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[t\sigma + i\left(\frac{x-\mu}{\sigma}\right)\right]^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dt$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[t\sigma + i\left(\frac{x-\mu}{\sigma}\right)\right]^2}\,dt \quad (1)$$

Normal dist^n $= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

put $t\sigma = z$

$\sigma\, dt = dz$

$dt = dz/\sigma$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z+i\left(\frac{x-\mu}{\sigma}\right)\right]^2} dz$$

$= 1$

Or $z \sim N\left(-i\left(\frac{x-\mu}{\sigma}\right), 1\right)$

Mean $= -\frac{i(x-\mu)}{\sigma}$ Var $= 1$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \times 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

so $\int_{-\infty}^{\infty}$ $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\sim N(\mu, \sigma^2)$

which the p.d.f. of Normal dist^n

Quest.^{n} Are (i) e^{-t^4} (ii) $e^{-t^2/2}$

(iii) $|\sin t|$ (iv) $t^{\log t}$

(v) e^{it} (vi) $\frac{1}{1+t^2}$ charater. funct^n?

Var. E[/ - E(x)]^2

If not why not? If so find the corresponding prob. functions f

Soln:→ $\phi(t) = e^{t^4}$

$$= 1 - \frac{t^4}{L1} + \frac{(t^4)^2}{L2} + \ldots \quad (1)$$

$\mu =$ coeff $\frac{it}{L1}$ in $\phi(t) = 0$

Var. = coeff $\frac{(it)^2}{L2}$ in $\phi(t) = 0$

$\because$ Var. $= E[e^{itx} x - E(x)]^2$

$= E(x-\mu)^2$

Or $E(x-\mu)^2 = 0$

$\sum (x-\mu)^2 p(x) = 0$

Or $(x-\mu)^2 = 0$

$\boxed{x = \mu}$

with certainity that $p(x=\mu) = 1$

$p(x \neq \mu) = 0$

$\phi_x(t) = E[e^{itx}]$

$\sum e^{itx} p(x)$

$\because E(x) = \sum x p(x)$

$= (x=\mu) p(x=\mu)$

$+ (x \neq \mu) p(x \neq \mu)$

$e^{i\theta} = 1$ $\qquad \therefore \phi(-t) \leq 1$

$= e^{it\mu} P(X=\mu) + e^{it(\neq \mu)} P(X \neq \mu)$

$= e^{it\mu} \cdot 1 + 0$

$= e^{it\mu} = 1 \neq e^{-t^4}$

$\Rightarrow e^{-t^4}$ is not ch. fuctn

(ii) $\phi(t) = e^{-t^2/2}$

is the ch. fuctn of S.N.V. $\sim N(0,1)$

(iii) $\phi(t) = |\sin t|$

Some standard result

(i) ~~$\phi(t) =$~~ $\phi(0) = 1$

(ii) $\overline{\phi(t)} = \phi(-t)$

(iii) $\phi(t) \leq 1$

(iv) $\phi(t)$ is uniformly cont. fuctn of t.

$\phi(0) = |\sin 0| = 0 \neq 1$

i.e. its not a ch. fuctn