From FA Smith, Applied Radiation Physics

CHAPTER 2

INTERACTIONS of CHARGED PARTICLES

2.1 Introduction

When ionizing radiation falls on a material, it is necessary to be able to answer three questions for a full understanding of the consequences. These are :

- how far into the material does the radiation penetrate?
- how much energy is deposited within the material?
- what secondary radiation or particles are emitted as a result?

The probability that an interaction takes place is described by its cross-section σ. This is a parameter which depends on both the type of interaction and the energy of the radiation. It describes the effective area which the entity (electron, nucleus, atom, molecule..) presents to the radiation in units of barn (1 barn = 10^{-28} m^2). Accordingly, each radiation will have a number of energy-dependent partial cross-sections. These can be segregated according to whether they are absorption, inelastic scattering, elastic scattering or radiative in character. In most circumstances the partial cross-sections *i* add algebraically to give the total, so that :

$$\sigma_{total} = \sum_i \sigma_i$$

In others – *e.g.* in neutron scattering – it is sometimes the amplitudes and not the intensities of the scattered waves which must be added. This means that the phases of the component waves have to be considered, leading to a more complex description of the total cross-section (see Chapter 4).

An important characteristic of charged particles which distinguishes them from photons and neutrons is that their penetration into a material cannot be described by an exponential function. Although there is a finite probability that a photon, however low in energy, can penetrate to an infinite depth, this is not the case for a charged particle. There is always an ultimate depth beyond which a charged particle will not reach. This leads to the concept of an energy-dependent Stopping Power of a medium towards a charged particle. It is defined as the average energy loss per unit path length traversed. Associated with this is the concept of Range, about which great care must be taken in order to distinguish between the various definitions.

For example, if the cross-sections for absorption, elastic scatter and inelastic scatter at a certain energy are σ_a, σ_s and σ_{inel}, the probability that the collision will be an inelastic one is $\sigma_{inel}/(\sigma_a+\sigma_s+\sigma_{inel})$.

Estimates of the mean free path between collisions, and the probability of energy loss in each one, can be made in a Monte-Carlo type of calculation. Here, a random number generator and a large data-base of cross-sections is used to compile a statistically meaningful number of particle histories. The technique provides a powerful means of predicting the effects of radiation on different materials in different conditions. However, care must be taken to incorporate the correct probability distributions into the calculations. These and other relevant considerations can be found in more specialized texts [1].

2.2 Definitions of Range

Two approximations to the complete stochastic nature of charged particle interactions are provided by :

- the Continuous Slowing Down Approximation (csda),
- the Straight Ahead Approximation.

In the former, all energy-loss fluctuations are neglected and the particles are assumed to lose their energy continuously along their tracks at a rate given by the stopping power. Since the stopping power is assumed to be a smooth function of energy, the csda range, r_0, can be defined as the integral with respect to energy of the reciprocal of the stopping power.

$$r_0 = \int_0^{T_0} \frac{1}{dE/dx}\, dE \tag{2.1}$$

The limits in Eq.(2.1) show that the particle is assumed to come to rest after slowing down from energy T_0. It should be noted that the csda range refers only to interactions which result in energy loss. It does not include any possibility of multiple scatter without energy loss, or thermal diffusion either before, or after, neutralization. For this reason, r_0 is always smaller – sometimes significantly so – than the mean value of the path lengths actually travelled.

Any departure from linearity is discussed in terms of a Detour Factor. This relates the csda range with the mean penetration depth (sometimes called the mean projected range) along the original direction of the particle, z_{av}, and is always <1. Thus, the Detour Factor, d is given by :

$$d = \frac{z_{av}}{r_0} \tag{2.2}$$

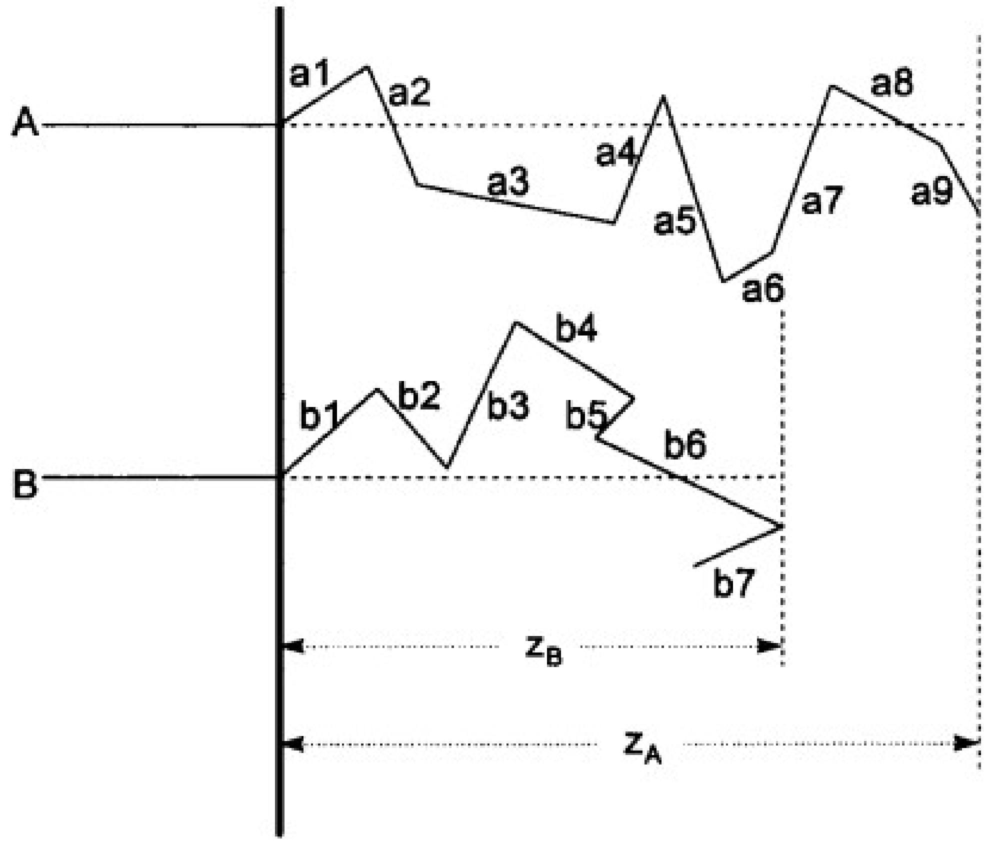

Fig.(2.1) Schematic of two charged particle tracks at the same incident energy. In the csda approximation the range, r_o, would correspond to the mean path length of a large number of such tracks, i.e. the mean of a1+..+a9, b1+..+b7, etc. so long as the individual interaction sites were energy loss events and there was no multiple scatter. With an increasing amount of multiple scatter, the average path length becomes larger than r_o.
The Detour Factor specified for track B is z_B /(b1+b2+b3+b4+b5+b6+b7).

From an experimental point of view, the determination of a quantity which reflects the penetrating power of an energetic charged particle can be carried out in two ways :

- by measuring the fraction of particles that are transmitted through increasing thicknesses of the medium,
- by measuring the depth-dependence of the response of a detector actually in the medium.

The parameters yielded by these two methods have slightly different meanings.

2.2.1 *The transmission method*

The decrease of the transmitted fraction of particles N_x/N_0 as x increases depends on the geometry in an experiment of this type. This is because the transmitted beam experiences a larger divergence as the thickness x of the absorber increases. The transmitted fraction is therefore a combination of particles stopping completely through energy loss, and particles which have been scattered out of the beam. The latter is determined by the position of the detector.

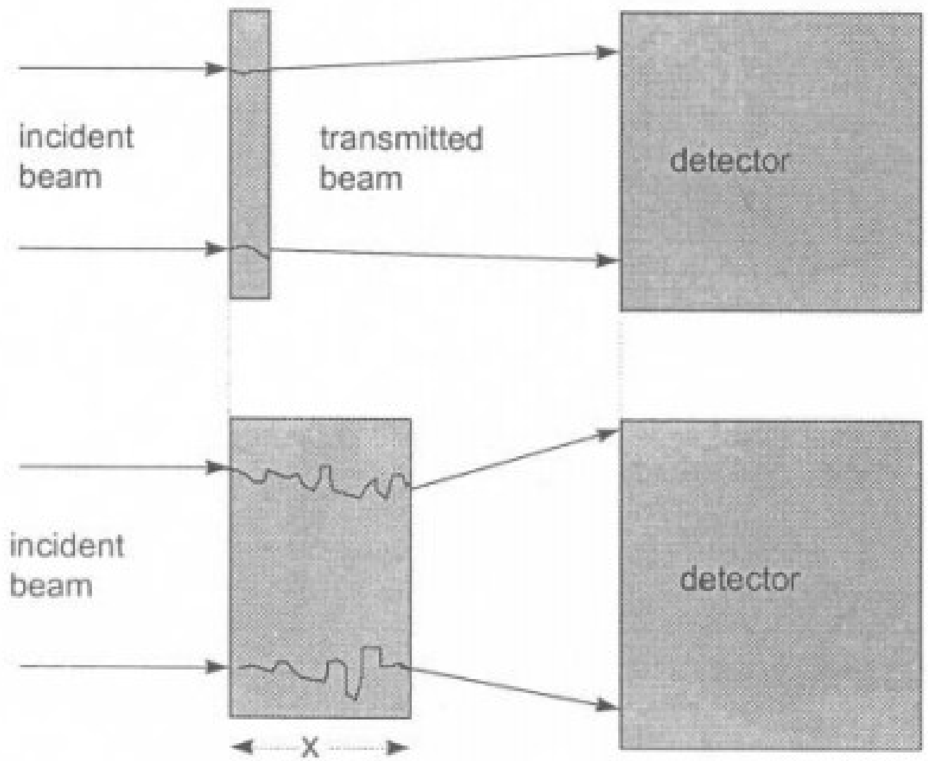

Fig.(2.4) The determination of the penetration, and hence the mean range, of a charged particle in a medium can be obtained by counting the number of particles N_x transmitted through a thickness x of the medium.

If such a transmission experiment were correctly performed, the number of particles N_x transmitted through thickness x, normalized to the number of incident particles N_0, is given by :

$$\frac{N_x}{N_0} = 1 - \int_{-\infty}^{x} \frac{1}{\alpha\sqrt{\pi}} \exp\left(-\frac{(x-R)^2}{\alpha^2}\right) dx \qquad (2.3)$$

In Eq.(2.3) the integration of the normal distribution in the second term proceeds up to thickness *x*, [4]. It is expressed in terms of two parameters, the range straggling parameter α and the mean range parameter *R*. The magnitude of *R* is defined as the thickness *x* for which $N_x/N_0 = ½$ and is determined by the energy of the particle. The width of the distribution of ranges, *i.e.* the range straggling, determines the magnitude of α. A light particle, which is subjected to large fluctuations in direction during its passage through the material, will have a large straggling parameter with the converse for a heavy particle.

In addition to range straggling, there is also the phenomenon of energy straggling. This describes the variations in the energy of a particle after it has travelled through a given thickness of material.

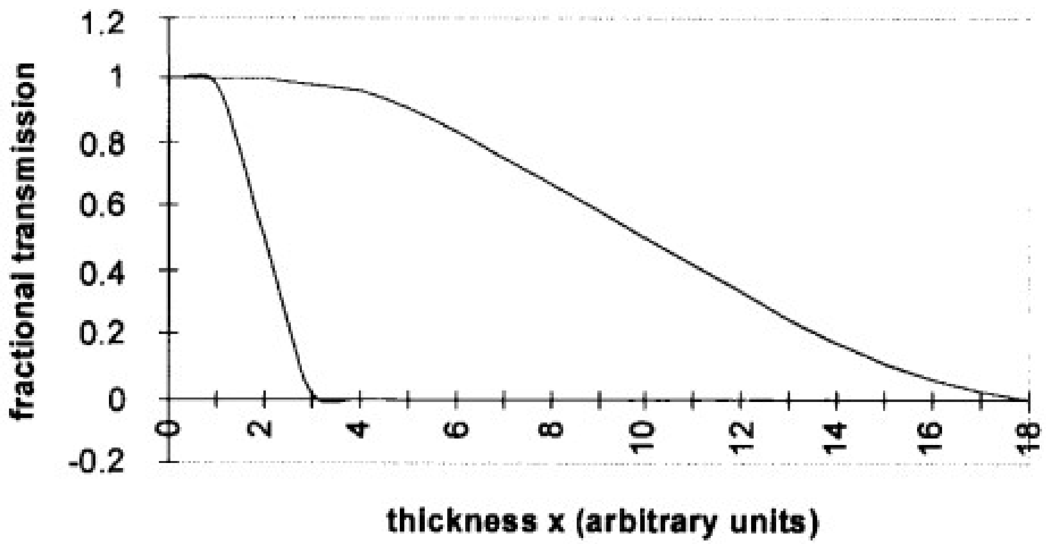

Fig.(2.5) A schematic representation of the percentage transmission of a charged particle through thickness *x* of absorbing material. The left-hand curve represents small values of *R* and α , e.g. a low energy alpha-particle, and the right-hand curve large values of *R* and α, e.g. a high energy electron. The mean ranges are 2.0 and 10.0 thickness units respectively.

2.2.2 The depth-dependence method

This method is used routinely in radiotherapy physics for two purposes. Firstly, to determine the mean range of the radiation and secondly, to achieve some measure of its quality (*i.e.* the spectral distribution). This is done largely through the measurement of the Tissue Phantom Ratio (TPR) which will receive more detailed attention in Chapter 9.

A beam of electrons is incident on the top surface of a tank of water in which a small-volume detector can move along the central beam axis, Fig.(2.6). The detector measures the energy deposited in its sensitive volume. For this reason the depth-dependence of the signal is called a Depth:Dose Distribution.

Fig.(2.7) differs from Fig.(2.5) in two important respects. Firstly, the absorbed dose when z is small is less than the maximum, D_m, because of the effect of Build-Up (see Chapter 7). Secondly, for large *z*, the dose does not become zero. This is due to the presence of *bremsstrahlung* radiation produced in the beam-defining collimators. The intersection of the line through the inflection point of the sigmoid portion of the curve (at R_{50}), and the straight line defining the *bremsstrahlung* tail defines the Practical Range, R_p. R_{50} corresponds to the mean range in Fig.(2.5). R_{ex} is the Extrapolated Range given by the extrapolation of the sigmoid portion onto the z-axis.

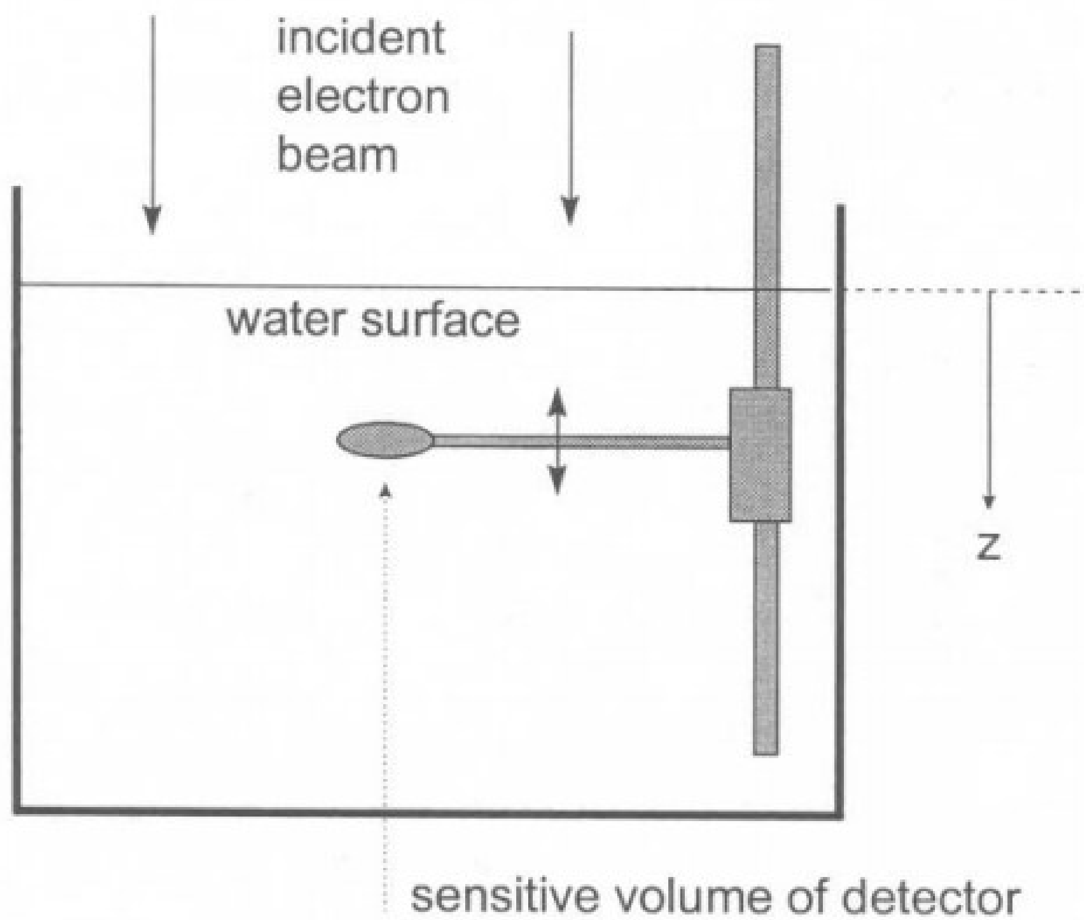

Fig.(2.6) Determination of the penetration of an electron beam in a tank of water. A (waterproofed) detector measures the ionization recorded in the sensitive volume as a function of depth, *z*, in the water.

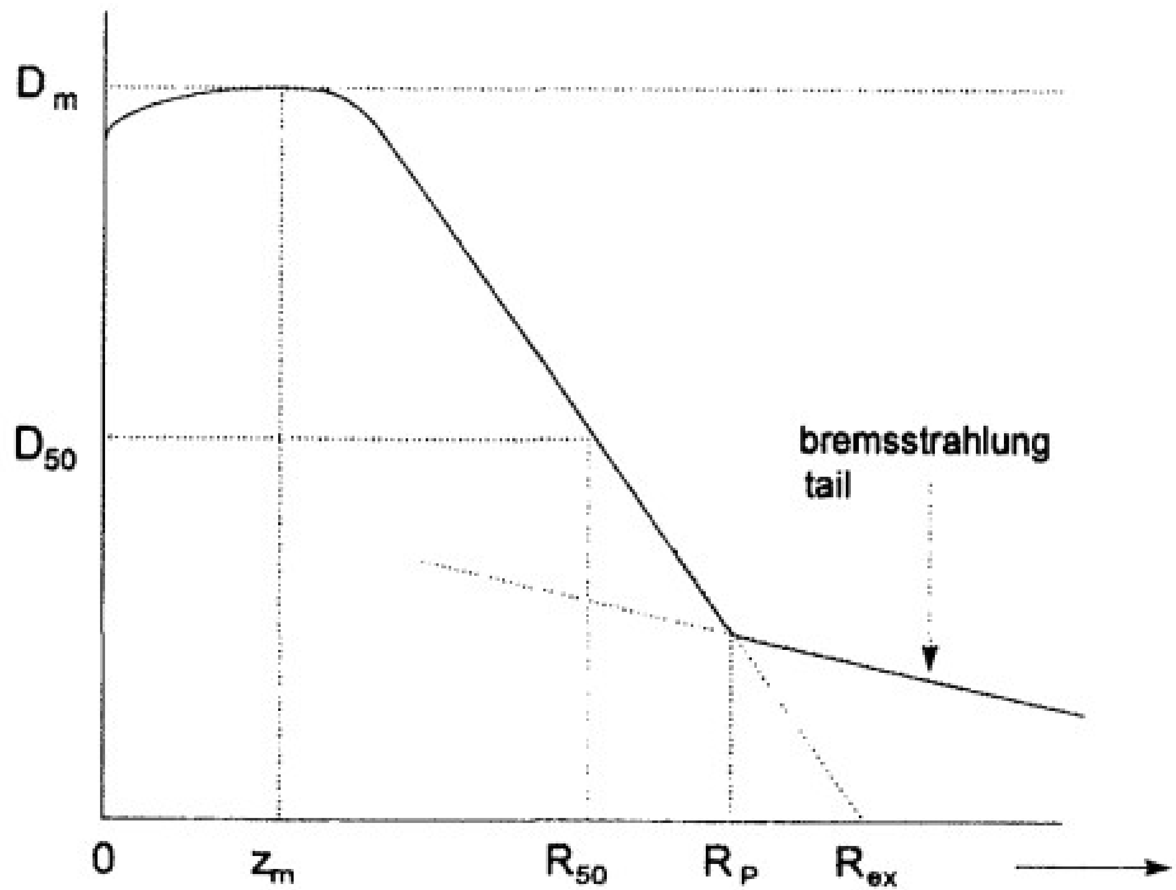

Fig.(2.7) The Depth Dose response of the detector in Fig.(2.6) as a function of depth *z*, in the irradiation of water by a beam of high energy electrons.

The following is a summary of the various parameters which relate to the penetration of a charged particle into a medium [5] :

r_0 – the csda range. This is approximately equal to the total path length, but can never be greater than it.

z_{av} – the mean projected range (or mean penetration depth).

mean path length. This is the summation of the individual distances travelled by the particle between energy loss events. It is slightly larger than r_0 because of the effects of multiple scatter and thermal diffusion.

d – the detour factor (z_{av}/r_0). This gives a measure of the non-linearity of the track of the particle ($d < 1$).

R – the mean range. In Fig.(2.1) this is z_{av}. In Fig.(2.7) it is R_{50}. In the experimental determination which uses the method of Fig.(2.6), there is also the Mean Practical Range R_p and the Mean Extrapolated Range R_{ex}. When there is no background or *bremsstrahlung* tail to the curve in Fig.(2.7), then $R_p = R_{ex}$.

A number of empirical expressions have been proposed to relate the mean practical range, R_p, in water with the incident electron energy. As a general rule of thumb, the mean practical range in cm is half the incident energy in MeV. For example :

In the energy range $0.01 \leq E \leq 3$ MeV, for $n = 1.265 - 0.0954 \ln E$

$$R_p = 0.412E^n \tag{2.4}$$

In the energy range $2.5 \leq E \leq 20$ MeV,

$$R_p = 0.530E - 0.106 \tag{2.5}$$

A useful source of range data for heavy charged particles is provided by the calculations of Barkas and Berger [6] for protons in water. Fig.(2.9) expresses the normalized range against energy/atomic mass unit.

2.3 Types of Charged Particle Interaction

Charged Particle Interactions fall into three broad groups :

(1) interactions with the individual electrons of atoms or molecules in the material. These interactions are by far the most important, and lead to excitation and/or ionization of the atoms or molecules.

- A collision is inelastic if the electron receives energy sufficient either to excite it into a higher bound state of the atomic system, or to eject it into an unbound state.
- If the amount of energy transferred is less than the smallest energy difference of the atomic or molecular levels (*i.e.* a sub-excitation collision) then the collision is regarded as an elastic scatter process in which energy and momentum are conserved. Clearly, the struck electron cannot be regarded as being isolated in these cases.

(2) interactions with nuclei. When the incoming particle is heavy (its mass is large compared to that of the electron), these include :

- nuclear reactions -- in which the particle is first absorbed into the nucleus with the formation and subsequent decay of a compound nucleus (treated further in Chapter 8),
- nuclear (Potential) or Coulomb (Rutherford) scattering which lead to the phenomenon of Multiple Scattering. When the incident charged particle is an electron however, the inelastic scattering from the strong Coulomb field surrounding the nucleus gives rise to *Bremsstrahlung* radiation. This accounts for a significant contribution to the energy loss of electrons at high energies.

(3) interactions with the whole Coulomb field surrounding an atom. Here, the interaction is between the incident particle and the coupled system of nucleus plus orbiting electrons. This type of interaction occurs only for low energy incident particles, either :

- those below the excitation potential of the target atoms (molecules) or
- very heavy particles with low velocities. This type of collision is often, but inappropriately, called a nuclear collision.

Within each of these three groups, the outcome of a collision is determined by a further three parameters :

- the velocity of the collision, v,
- the distance of closest approach of the participants, r_{min}, and
- the range of the potential which governs the interactions between the incident particle and the target.

Table (2.1) Qualitative categorization of different types of collision.

Type of collision	soft ⇒ "isothermal"	hard ⇒ "adiabatic"
duration of collision	long	short
distance of closest approach	large	small
ion velocity/ orbital electron velocity	small	large
energy transferred	low	high

Table (2.1) shows how these three parameters lead to a distinction between soft or hard collisions. A collision is soft or hard when the distance of closest approach is either large or small compared with a characteristic range of the interaction potential. Similarly, fast or slow collisions are those characterized by projectile velocities which are high or low compared with the orbital velocity of the electrons in the target atom. If the collision time is short compared with a characteristic lifetime of the target system, we have an adiabatic collision with large energy transfer. As the collision time increases, there is a decrease in the amount of energy that can be transferred to the system.

2.4 Energy Transfer in an Elastic Collision - Classical Theory

The most likely outcome of the transport of a fast charged particle through a medium is the transfer of energy, via Coulomb interactions, to the electrons of the medium. In the non-relativistic condition, the kinetic energy of the incident charged particle,

T, is much smaller than its rest mass, Mc^2.

We therefore consider the collision between an incident particle of mass *M*, charge *ze* and velocity *v* with an electron of mass m_0, charge *-e*. The target electron is initially stationary in the Laboratory (L) System of coordinates. As a result of the collision, energy *E* is transferred from *M* to m_0, the latter being ejected at an angle ϑ with respect to the initial direction of mass *M*, [4].

In the L system, the total momentum is *Mv* and the Centre of Mass moves to the right in Fig.(2.10) with velocity $U=Mv/(M+m_0)$. In the CM system, the incident particle has a velocity before the collision of :

$$v-\frac{Mv}{M+m_0}=\frac{vm_0}{M+m_0}$$

Since the linear momenta are equal and opposite before and after the collision, the velocities of the two particles are in the inverse ratio of their masses. The electron velocity after the collision is therefore :

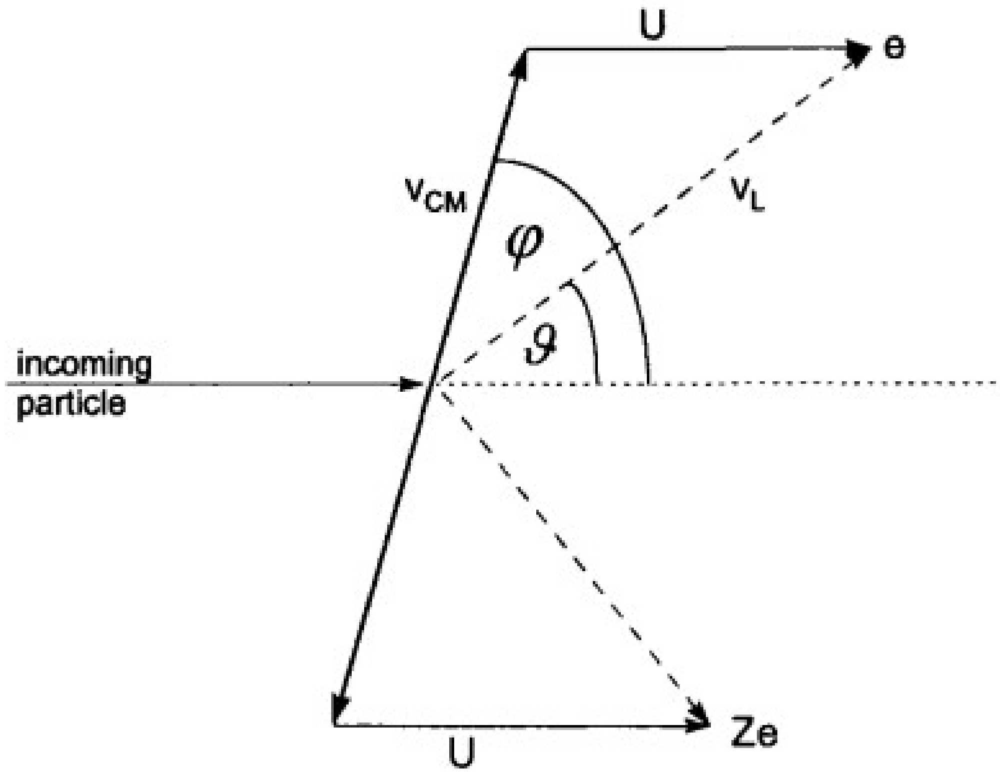

Fig.(2.10) Energy transfer between a mass *M* and an electron initially at rest in the Laboratory frame of reference. After the collision, the velocities of the struck electron in the L and Centre of Mass (CM) systems are v_L and v_{CM}. These make angles of ϑ and φ respectively with the initial particle direction. *U* is the velocity of the Centre of Mass.

$$v_{CM} = \frac{Mv}{M + m_0}$$

giving an isoceles triangular relation between v_L, v_{CM} and U. The velocity of the electron in the L system is then given by :

$$v_L = 2\frac{Mv}{M + m_0}\cos\vartheta$$

where ϑ is the ejection angle of the electron with respect to the incoming particle trajectory. The energy of the recoil electron is :

$$E = \frac{1}{2}m_0\left(\frac{2Mv}{M + m_0}\right)^2\cos^2\vartheta$$

If the ejection angle of the electron is φ in the CM system then, because the triangle is isoceles, we have $\varphi = 2\vartheta$. So :

$$E = 2m_0\frac{M^2v^2}{(M + m_0)^2}\frac{(1 + \cos\varphi)}{2}$$

For maximum energy transfer, $\cos \varphi \rightarrow 1$. When, in addition, the incoming particle is heavy so that $M >> m_0$, we have the condition that the maximum energy that a heavy particle can transfer to a stationary electron is :

$$E_{max} = 2m_0v^2 \tag{2.6}$$

When the incident particle is an electron, we have $E_{max} = m_0v^2/2$. Note that m_0 is the mass of the (struck) electron, and v is the velocity of the incoming particle of mass M. When the incident particle becomes relativistic, we use $\tau = T / Mc^2$ to express the ratio of the kinetic energy of the particle to its rest energy. Eq.(2.6) is then more correctly expressed as :

$$E_{max} = \frac{2\tau(\tau + 2)m_0c^2}{\left[1+2(\tau+1)(m_0/M)+(m_0/M)^2\right]} \tag{2.7}$$

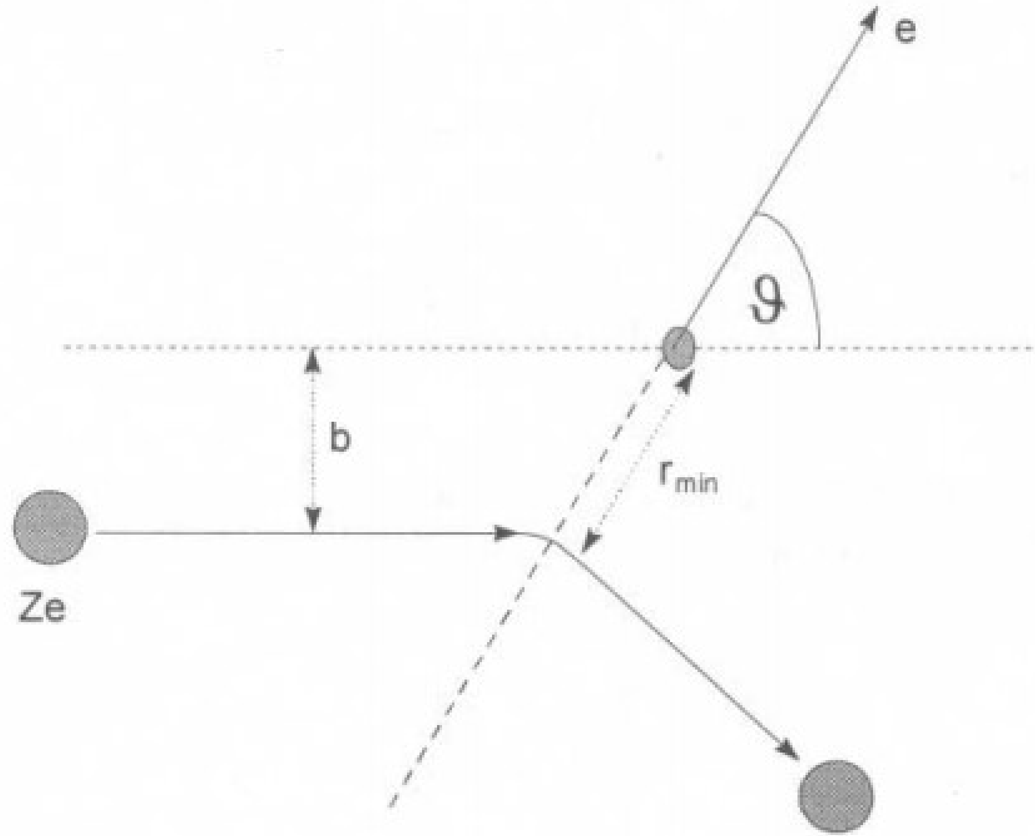

Fig.(2.11) The meaning of the impact parameter, *b*, and the collision diameter, r_{min}. ϑ is the scattering angle of the struck electron in the L system.

It is now necessary to consider the two distances which define a classical collision. The first is the impact parameter b which gives the separation of the two trajectories. The second is the collision diameter r_{min} which gives the minimum separation distance between the colliding particles, Fig.(2.11).

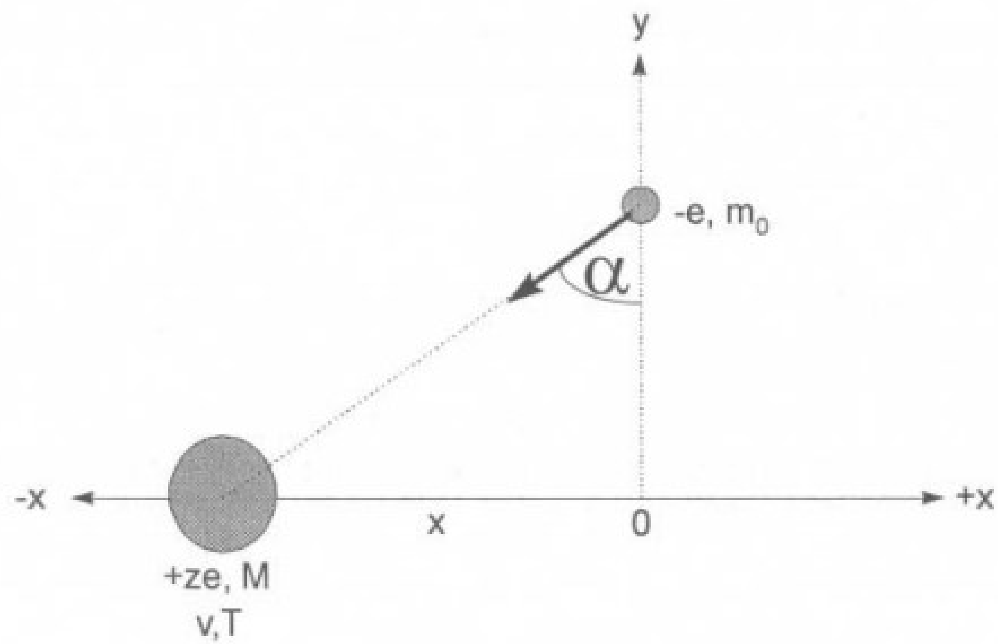

Fig.(2.12) The charged particle interaction between mass M and a free electron. The distance, x, goes from - to + infinity during the collision. The impact parameter, b, is the perpendicular distance between the trajectory and the target electron.

Two assumptions are made – firstly, that $M >> m_o$, so that the trajectory can be considered to be straight and secondly, that the electron is stationary before the collision. Fig.(2.12) represents the collision in terms of the kinetic energy of the projectile *T* and its impact parameter *b*.

The collision diameter, r_{min}, is related to *b* through the deflection angle in the L frame by :

$$b = \frac{r_{min}}{2} \tan \vartheta$$

When the electron is ejected at 45° in the L frame, the collision diameter is twice the impact parameter.

The attractive Coulomb force between the two particles is :

$$F = \frac{ze.e}{4\pi\varepsilon_0 (b / \cos\alpha)^2}$$

Since the x-component of F cancels over the duration of the collision, the remaining y-component is :

$$F_y = \frac{ze^2 \cos^3 \alpha}{b^2}$$

after omitting the permittivity term. The momentum transferred to the electron is :

$$q = \int F_y dt ,$$

where the duration of the collision, *t*, may be approximated as $b \tan \alpha / v$. Therefore $dt = (b/v) \sec^2 \alpha \, d\alpha$ and the momentum transfer becomes :

$$q = \frac{ze^2}{bv} \int_{-\pi/2}^{\pi/2} \cos \alpha . d\alpha = \frac{2ze^2}{bv}$$

The energy transfer for a given value of *b* is then :

$$E = \frac{q^2}{2m_0} = \frac{2z^2e^4}{m_0 b^2 v^2} \tag{2.8}$$

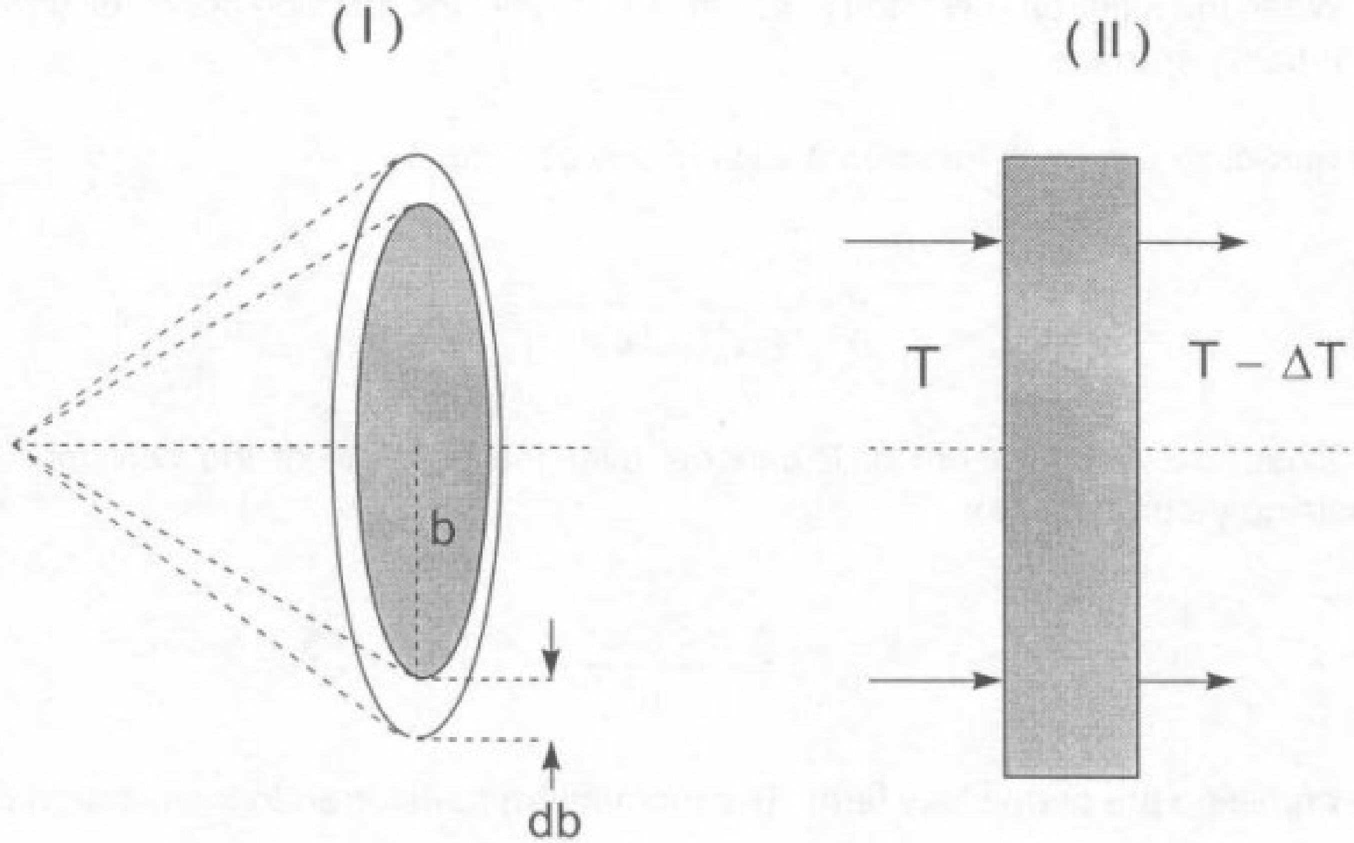

Fig.(2.13) (I) The definition of differential cross-section, $d\sigma$, for an impact parameter lying between b and b+db, is simply the area of the annulus of radius b and thickness db.
(II) We consider the energy loss ΔT in material of thickness Δx, atomic number Z, containing N atoms cm^{-3}.

Note the reciprocal relationship between the energy transferred and the square of the impact parameter. Although b goes to zero in a head-on collision, we cannot have $E \rightarrow \infty$, since Eqs.(2.6) and (2.7) give the maximum (finite) energy transfer possible in such a case. Therefore, there is a non-zero value of the smallest impact parameter given by, [4] :

$$b_{min} = \sqrt{\left(\frac{2z^2e^4}{m_0 E_{max} v^2}\right)} = \frac{ze^2}{m_0 v^2} \tag{2.9}$$

When the struck electron is free, there is no lower limit to the energy transferred (and hence no upper limit to the impact parameter, b). For bound electrons, however, the initial velocity of the electron will not be zero but determined by its orbital period, $T_0 = 1/\omega$, where ω is the orbital angular velocity. If we specify a characteristic time of the collision, $\tau_c = b/v$, such that $\tau_c < T_0$ then we have an adiabatic collision which results in energy transfer. If, on the other hand, $\tau_c > T_0$, the collision is deemed to be "soft", causing a deformation and relaxation of the electron orbit without any transfer of energy. The maximum impact parameter is given by the condition $\tau_c = T_0$ which yields $b_{max} = v/\omega$ and :

$$E_{min} = \frac{2z^2e^4\omega^2}{m_0v^4} \tag{2.10}$$

Eqs.(2.6) and (2.10) now give the maximum and minimum limits within which energy transfer from incident (heavy) ion to (light) electron can take place.

The total energy loss ΔT ($<<T$) is made up of a large number of individual collisions in which the amount of energy transferred, E, depends on the value of b. If dn is the number of collisions in which the energy transferred is between E and $E+\Delta E$, and the incident particle flux sees υ electrons per unit area, then :

$$\Delta T = \int E\, dn = \upsilon \int_{E_{min}}^{E_{max}} E\, d\sigma$$

For a material of thickness Δx, atomic number Z and atomic density N (atoms cm^{-3}), we have $\upsilon = N\, Z\, \Delta x$. Further, from Fig.(2.13), the interaction cross-section is by definition, $d\sigma = 2\pi b\, db$. By differentiating Eq.(2.8), we get the differential cross-section in units of cm^2/electron as :

$$d\sigma = 2\pi b\ db = \frac{2\pi z^2 e^4}{m_0 v^2} \frac{dE}{E^2} \tag{2.11}$$

This is an important result. The probability of an energy loss event taking place is inversely proportional both to the square of the velocity of the incoming particle and to the square of the amount of energy transferred in that collision.

The Stopping Power of the material towards the progress of the particle is then:

$$\frac{\Delta T}{\Delta x} = \frac{NZ 2\pi z^2 e^4}{m_0 v^2} \int_{E_{min}}^{E_{max}} \frac{dE}{E} = \frac{4\pi NZz^2 e^4}{m_0 v^2} ln\left[\frac{m_0 v^3}{ze^2 \omega}\right] \tag{2.12}$$

The classical description in Eq.(2.12) is unable to go beyond the case of a heavy particle interacting with a single electron orbiting an isolated atom. It does not consider :

- any details of those collisions which result in excited atoms or molecules,
- the differences between atoms in the isolated (gaseous) and condensed states,
- collisions approaching relativistic velocities,

- collisions at very low velocities where the incident particle sees the complex scattering potential due to the whole atomic electron cloud,
- interactions with the nucleus.

2.5 Stopping Power of a Charged Particle - the Bethe Formula

A quantum mechanical approach starts by using the Born approximation. This requires that the energy transferred is small compared with the energy of the projectile, when only the first order terms in the interaction energy are needed. In this case, the Bethe expression for energy loss of a heavy charged particle at non-relativistic velocities (the so-called electronic or collisional loss) is based on the following, [5] :

- it requires that $\frac{z\,Z}{\beta} << 1$. For the Born approximation to be valid, the velocity of the incoming ion or electron, *v*, must be much greater than the orbital velocity of the struck electron, *u*. If *u* is the Bohr-orbit velocity of a K-shell electron in the medium of atomic number *Z*, then :

$$\frac{u}{v} = \left(\frac{Z}{137\beta}\right) << 1$$

- it sums the contributions from hard and soft collisions,
- it introduces the mean excitation energy, I. This is a weighted average of all possible energy transfers from the incoming particle to an electron of the medium. It includes excitations as well as ionizations,
- it introduces the cut-off energy, E_c, to distinguish between hard and soft collisions.

The Bethe expression for the electronic contribution to the mass stopping power of a particle with velocity $\beta = v/c$ in a medium of density ρ is :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right)_{electronic} = \frac{4\pi r_e^2 m_0 c^2}{\beta^2}\frac{1}{u}\frac{Z}{A}z^2\left[ln\left(\frac{2m_0c^2\beta^2\gamma^2}{I}\right) - \beta^2\right] \tag{2.13}$$

In Eq.(2.13), $u = 1.6606 \times 10^{-24}$ g is the atomic mass unit, $r_e = e^2/m_0c^2 = 2.818 \times 10^{-13}$ cm is the classical electron radius and $\gamma^2 = 1/(1-\beta^2)$. The expression is valid only for projectile velocities which are large compared to the velocities of atomic electrons.

2.5.1 *Mean excitation energy*, I

The stopping power is only logarithmically dependent on the mean excitation energy and is therefore rather insensitive to changes in I. Nevertheless, the determination of I from measured stopping power data in different materials is the most direct method. More accurate methods involve the use of optical oscillator strengths (derived from photo-absorption cross-sections) or dielectric response functions, but fewer data are available.

The oscillator strength f_i is used to denote the participation of the ith electron in the energy loss process. For a given atom, we have :

$$ln\,\mathrm{I} = \sum_i f_i\, ln\, E_i \tag{2.14}$$

where E_i represents the energy transferred to the ith electron. This can be an electronic excitation of the atom or molecule or an excitation into the continuum, *i.e.* an ionization.

A comprehensive tabulation of I values is given [3]. For high atomic number materials, $\mathrm{I} \approx 10\ Z$ (eV). Other empirical expressions have been given by Barkas and Berger [6] :

$$I(Z) = 12Z + 7 \text{ eV} \qquad \text{for } Z \leq 13$$

$$I(Z) = 9.76Z + 58.8Z^{-0.19} \text{ eV} \qquad \text{for } Z \geq 13$$

For mixtures and compounds containing N_w atoms cm^{-3} of atom w having atomic number Z_w, use can be made of the Bragg Additivity Rule :

$$\ln I = \frac{\sum_w N_w Z_w \ln I_w}{\sum_w N_w Z_w} \tag{2.15}$$

2.5.2 *The cut-off energy, E_c, and restricted stopping power*

The use of a cut-off energy makes it possible to restrict the energy deposited in a medium to events which are localized. The extent of the localization is determined by the size of the cut-off energy. Energy depositions which result in the production of very energetic secondary electrons can therefore be excluded and treated as separate events. Eq.(2.13) can be rearranged to include the limiting cut-off energy as follows :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right) = \frac{4\,\pi\, r_e^{\,2} m_0 c^2}{\beta^2}\frac{1}{u}\frac{Z}{A}z^2\left[ln\left(\frac{2m_0c^2\beta^2\gamma^2}{\mathrm{I}}\right) - \beta^2\right]$$

$$= \frac{2\,\pi\, r_e^{\,2} m_0 c^2}{\beta^2}\frac{1}{u}\frac{Z}{A}z^2\left[2\,ln\left(\frac{2m_0c^2\beta^2\gamma^2}{\mathrm{I}}\right) - 2\beta^2\right]$$

$$\left[ln\left(\frac{\left(2m_0c^2\beta^2\gamma^2\right)^2}{\mathrm{I}^2}\right) - 2\beta^2\right]$$

$$\left[ln\left(2m_0c^2\beta^2\gamma^2\right)^2 - 2\,ln\,\mathrm{I} - 2\beta^2\right]$$

$$\left[ln\left(2m_0c^2\beta^2\gamma^2E_c\right) - 2\,ln\,\mathrm{I} - 2\beta^2\right]$$

where E_c gives the limiting energy cut-off. The maximum energy that can be transferred to an unbound electron is given in Eq.(2.6) for a heavy incident particle. For a smaller energy transfer limit, determined only by the velocity of the incoming ion, we have :

$$E_c = 2m_0c^2\beta^2\gamma^2$$

The square bracket term is now divided by 2 to give :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right)_{E_c} = \frac{4\,\pi\, r_e^{\,2} m_0 c^2}{\beta^2}\frac{1}{u}\frac{Z}{A}z^2\left[\frac{1}{2}\ln\left(\frac{2m_0c^2\beta^2E_c}{\left(1-\beta^2\right)}\right) - \ln \mathrm{I} - \beta^2\right] \tag{2.16}$$

Eq.(2.16) now gives the stopping power restricted to energy transfers smaller than E_c. Two conditions can then be specified :

- When E_c is larger than the largest binding energy of the atomic electrons in the medium. In this case E_c is much larger than the K-shell binding energy of the highest atomic number constituent. Energy losses by the incoming particle which are smaller than E_c must therefore result from collision impact parameters which are large compared to the atomic dimensions.
- When an energy transfer is larger than E_c. In this case the energy transfer must lie in the range E_c to E_{max}, Eq.(2.6). It must be the result of a collision at such a small impact parameter that the target electron can be considered to be free and at rest.

The factors before the square brackets in Eq.(2.16) give the gross features of the energy loss, having the value :

$$\frac{4\pi\ r_e^{\,2} m_0 c^2}{u}\left(\frac{z^2 Z}{\beta^2 A}\right) = 0.307\left(\frac{z^2 Z}{\beta^2 A}\right) MeV\ cm^2\ g^{-1} \tag{2.17}$$

Using Eq.(2.17) and $T_{proton} = \beta^2 \times 4.697 \times 10^5$ keV, the effect on the total mass stopping power of different cut-off energies E_c can be illustrated in Fig.(2.14) for a non-relativistic proton traversing carbon.

Note that no data points appear in Fig.(2.14) for the lowest incident energies and the lowest cut-off limits. This is because the energy transferred is so small that the sum of the two logarithmic terms in Eq.(2.16) becomes negative.

The quantity in the square brackets of Eq.(2.16) is frequently replaced by the dimension-less Stopping Number, *L,* to account for the fine details of the stopping process.

$$L(\beta) = L_0(\beta) + zL_1(\beta) + z^2 L_2(\beta) \tag{2.18}$$

The principal term in Eq.(2.18) describes the effectiveness of the incident particle in ionizing a target atom or molecule. It contains the square brackets of Eq.(2.16) together with two correction terms. These are :

- the shell term *C/Z*. This accounts for the effects at low incident particle energies when the incoming velocities become smaller than the velocities of K, L,.. – shell electrons of the target atom. When the ion velocity is smaller than that of a K–shell electron, the latter no longer contributes to the stopping process and the stopping power is reduced by the negative term $-(C/Z)_K$. Similar terms account for the effects in L–shell, and even less tightly bound, electrons. The shell term increases as the *Z* of the material increases and as the velocity of the incident particle decreases [3].
- the density term $\delta/2$. At high incident velocities the fast moving charge polarizes the atoms of the medium in such a way as to reduce the electromagnetic field acting on the particle. Consequently a decrease is again seen in the stopping power.

The principal term in the stopping number is then :

$$L_0(\beta) = \frac{1}{2} ln\left(\frac{2m_0c^2\beta^2E_c}{1-\beta^2}\right) - \beta^2 - ln\, I - \frac{C}{Z} - \frac{\delta}{2}$$

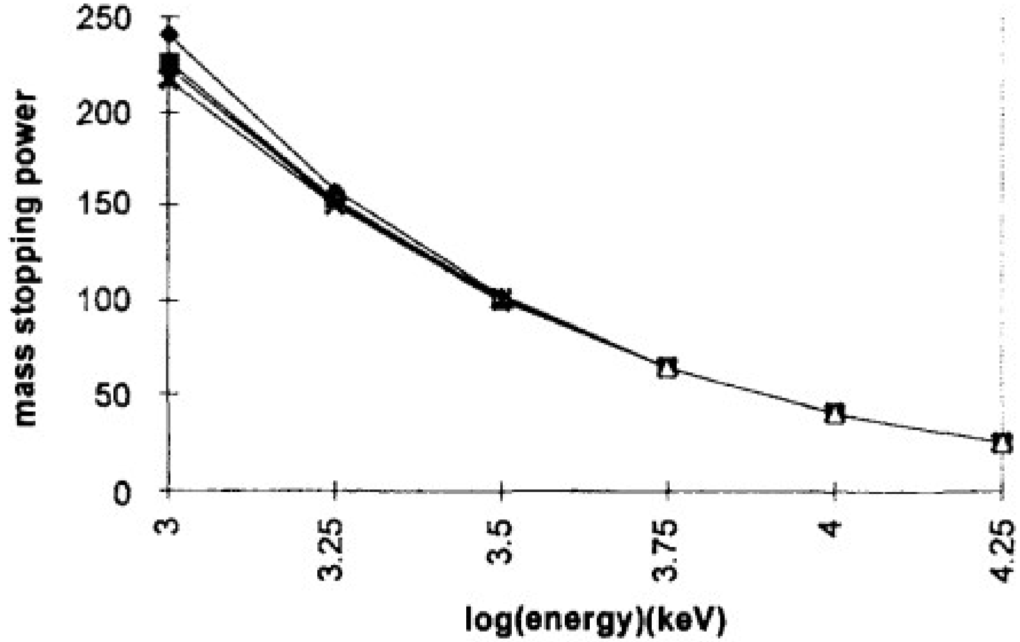

Fig.(2.15) The effects of Shell and Barkas correction terms on the mass stopping power of protons in carbon (MeV cm^{-2} g^{-1}). The cut-off energy E_c is given by Eq.(2.6) (i.e. the maximum value): ◆ Eq.(2.16); ■ Eq.(2.16) with K-shell correction term added; ▲ Eq.(2.16) with K-and L-shell terms added; ✖ Eq.(2.16) with K- and L-shell and Barkas terms added.
The Bloch $z^2L_2(\beta)$ and the density $\delta/2$ terms are insignificant at these incident energies. Data have been interpolated from [3].
These correction terms clearly become more important in reducing the mass stopping power, as given in Eq.(2.16), when the incident proton energy becomes smaller than 1 MeV.

The second term in Eq.(2.18), $zL_1(\beta)$, is known as the Barkas term. It contains the charge on the incident ion, z, raised to the first power. The term therefore changes sign on going from particle to anti-particle. It has the effect of reducing the stopping power of an electron, for example, compared with that of a positron at the same energy. Further reference may be made to ICRU-49 [3] and ICRU-37 [5].

The third term, $z^2L_2(\beta)$, is the Bloch term. It accounts for the departures from the first-order Born approximation. It is inversely proportional to β^2, and has the effect of reducing the stopping power for low energy particles [3], [5].

Note that at low incident velocities, the electronic contribution to the mass stopping power of a charged particle is reduced when :

- there is a reduction in the cut-off energy E_c below that given by Eq.(2.6), see Fig.(2.14),
- the effects of the Shell and Barkas correction terms become important, see Fig.(2.15).

2.6 Theoretical Description for Light Charged Particles

An incoming electron has the same mass as the struck electron in an electronic energy loss process and is indistinguishable from it. The stopping power is therefore considered to apply to the faster of the two particles that emerge from the collision.

Eq.(2.11) gives the differential cross-section for the transfer of energy between E and $E + dE$ to a free electron. When the incoming particle is itself an electron we have $z = -1$. The differential cross-section for the incident electron to lose energy E to the struck electron, and emerge from the collision with energy $T - E$, is :

$$\frac{d\sigma}{dE} = \frac{2\pi e^4}{m_0 v^2} \frac{1}{E^2}$$

Because the incident and target electrons are indistinguishable, there is also the possibility that $T - E$ is transferred to the struck electron and the incident electron emerges with energy E. The total cross-section is the sum of these two possibilities:

$$\frac{d\sigma}{dE} = \frac{2\pi e^4}{m_0 v^2} \left[\frac{1}{E^2} + \frac{1}{(T-E)^2} \right]$$

$$= \frac{2\pi e^4}{m_0 v^2} \frac{1}{E^2} \left(\frac{T}{T-E} \right)^2 \left[1 - 2\left(\frac{E}{T}\right) + 2\left(\frac{E}{T}\right)^2 \right] \qquad (2.19)$$

In this case, the maximum energy transfer is now $T/2$ and the subsequent formulae for the stopping power therefore apply to energy transfers $E \leq T/2$. From Eq.(2.6) this gives $E \leq E_{max}/2$.

Just as in the case of heavy charged particles, a distinction is made between hard and soft collisions in the transition from a classical to a quantum mechanical description.

- a hard collision is one in which the energy transfer extends from the maximum ($T/2$) down to an arbitrary value E_H . This is large compared with the binding energy of the struck electron,
- in a soft collision, the energy transferred extends downwards from E_H to the minimum needed to produce either an excitation or an ionization of an atom or molecule of the medium.

The mass stopping power can therefore be given by :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right)_{electronic} = \frac{1}{\rho}\left(\frac{dT}{dx}\right)_{E<E_H} + \frac{1}{\rho}\left(\frac{dT}{dx}\right)_{E>E_H} \qquad (2.20)$$

For incident electrons, hard collisions will be infrequent because they result in the loss of almost half of the incident energy.

When Eq.(2.19) is corrected for relativistic, spin and exchange effects, the total cross-section for energy loss becomes the Møller cross-section. This applies to the collision of indistinguishable particles. When combined with Eq.(2.20) it gives :

$$\frac{1}{\rho}\frac{dT}{dx} = \frac{2\pi r_e^2 m_0 c^2}{\beta^2}\left(\frac{Z}{uA}\right)\left[2\, ln\left(\frac{T}{I}\right) + ln\left(\frac{2+\tau}{2}\right) + F^{\pm}(\tau) - \delta\right] \qquad (2.21)$$

In the comparison with Eq.(2.17), note the appearance of 2 instead of 4 in the pre-logarithmic term, and the fact that $z^2 = 1$ for an electron or positron. The function $F^{\pm}(\tau)$ is defined for an electron and a positron as :

$$F^{-}(\tau) = \left(1-\beta^2\right)\left[1 + \frac{\tau^2}{8} - (2\tau + 1)\, ln\, 2\right]$$

$$F^{+}(\tau) = 2\, ln\, 2 - \left(\frac{\beta^2}{12}\right)\left[23 + \frac{14}{\tau + 2} + \frac{10}{(\tau+2)^2} + \frac{4}{(\tau+2)^3}\right]$$

For non-relativistic electrons we have $\beta \rightarrow 0$ and $I << T << m_0c^2$. Eq.(2.21) then reduces to :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right)_{electronic} = \frac{4\pi e^4}{m_0 v^2}\left(\frac{Z}{uA}\right) ln\left(\frac{m_0 v^2}{I\sqrt{2}}\right) = \frac{2\pi e^4}{T}\left(\frac{Z}{uA}\right) ln\left(\frac{T\sqrt{2}}{I}\right) \qquad (2.22)$$

2.7 Interactions of Low Energy Electrons

Below ~2 keV, the Bethe-Bloch theory becomes inadequate for electrons, because the incident velocity becomes comparable to, and eventually smaller than, the Bohr orbital velocity.

At low energies it becomes more appropriate to consider energy loss in terms of the dielectric response of the medium. In this case the situation resembles that in electrostatics, when the effect of an external field is reduced if it induces a polarization of the medium. The extent to which this screening of the external field takes place is determined by the dielectric constant.

This principle is now generalised [8] to include :

- external fields which vary in both space and time – as in the case of a charged particle approaching an atom,
- a longitudinal electric field, $\Delta(r,t)$, which produces a charge density $\rho(r,t)$ in the medium at location r and time t,
- the assumption that the applied field is sufficiently weak that $\rho(r,t) \propto \Delta(r,t)$.

Equations which relate the internal field $E(r,t)$, the displacement field $\Delta(r,t)$ and the charge density $\rho(r,t)$ are Fourier-transformed in space and time. The resulting fields can then be expressed in terms of $\varepsilon(q,\omega)$, the wave vector and frequency-dependent dielectric constant. Thus :

$$E(q,\omega) = \frac{\Delta(q,\omega)}{\varepsilon(q,\omega)}$$

On a microscopic scale, the wave vector q and frequency ω denote the momentum transfer ($\hbar q$) and energy loss ($\hbar\omega$) arising from the passage of the charged particle. On the macroscopic scale, it is the rigidity of the electronic structure that determines the dielectric response. A comparison can therefore be drawn with the involvement of core and valence electrons in the energy loss of an electron as it travels through an insulating medium.

From the early consideration of interactions in a Fermi-Dirac electron gas [9], expressions have been found [10],[11] for :

- the inverse mean free path of an electron with initial energy E,

$$\mu(E) = \int_0^{0.583E} d(\hbar\omega)\,\frac{d\mu}{d(\hbar\omega)} \tag{2.23}$$

  In Eq.(2.23) the upper limit of integration accounts approximately for the influence of exchange between the incident electron and the electrons in the medium,
- the differential inverse mean free path with respect to energy loss:

$$\tau(E,\hbar\omega) = \frac{d\mu}{d(\hbar\omega)} = \frac{1}{\pi r_B E}\int_{q-}^{q+}\frac{dq}{q}\, Im\left[\frac{-1}{\varepsilon(q,\omega)}\right] \tag{2.24}$$

where the limits of integration of the momentum are given by

$\hbar q_{\pm} = \sqrt{2m_0}\left[\sqrt{E} \pm \sqrt{E-\hbar\omega}\right]$ for an energy loss between $\hbar\omega$ and $\hbar\omega + d(\hbar\omega)$.

$r_B = \hbar^2/m_0 e^2$ is the Bohr radius,

- the stopping power $S(E) = \int_0^{0.583E} d(\hbar\omega)\; \hbar\omega\; \frac{d\mu}{d(\hbar\omega)}$ (2.25)

2.8 Momentum Loss of Heavy Charged Particles

This contribution to the stopping power comes from the recoil of an atom of the medium following the Coulomb elastic scatter of the incident projectile. Although it is commonly referred to as nuclear collision loss, the process is not actually nuclear, but merely the scatter of an incoming charge *ze* by a target charge *Ze*. The potential between the two, when they are separated by distance r is :

$$V(r) = \frac{zZe^2}{r} F_s(r / r_s) \qquad (2.28)$$

Here $F_s(r/r_s)$ is the parameter which takes account of the screening of the target charge by the atomic electrons.

The screening length r_s can be expressed in a number of ways. Numerical calculations of the screening function for many different combinations of projectile and target have been used to yield universal expressions for α-particles and heavier ions. For example, using the Bohr radius $r_B = \hbar^2/m_0e^2$, we have :

$$r_s = 0.88534 r_B \left(z^{0.23} + Z^{0.23}\right)^{-1}$$

$$F_s(r / r_s) = 0.2\exp(-3.2r / r_s) + 0.5\exp(-0.9r / r_s) + 0.3\exp(-0.4r / r_s) + 0.3\exp(-0.2r / r_s)$$

An important aspect of the passage of positive ions through matter is the variation of charge state with energy. At each collision, a positive particle has a certain probability of picking up an electron and thereby reducing its positive charge state by one. At a subsequent collision there is a finite probability that the particle can lose this electron back to an atom of the medium. The ratio of these two probabilities, electron gain to electron loss, increases as the energy of the particle decreases. As a consequence, a positive ion ultimately becomes a neutral atom at the end of its range.

The steps necessary in the formulation of the nuclear collision stopping power from the scattering potential in Eq.(2.28) are now :

- the determination of the angle of scatter in the centre-of-mass system of co-ordinates using
$$\theta = \pi - 2\int_{r_{min}}^{\infty} \frac{1}{r^2} \frac{1}{\left[1 - V(r)\frac{M + M_t}{T\ M_t} - \frac{b^2}{r^2}\right]^{1/2}} b\ dr$$
where r_{min}, M and b have the same meaning as in section 2.4. M_t is the mass of the target atom. (Note that in this expression θ is the angle of scatter of the incident positive ion, whereas ϑ in section 2.4 is the angle of scatter of an electron in an ionization event),
- the determination of the elastic scattering cross-section. This is obtained from the numerical differentiation of the above relation between θ and b using
$$\frac{d\sigma_{el}}{d\Omega} \sin\theta = -b\frac{db}{d\theta}$$

The mass nuclear stopping power is then given by :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right)_{nuclear} = -2\pi N \int_0^\infty b \frac{db}{d\theta} E(\theta, T) db$$

where $E(\theta,T) = E_m \sin^2(\theta/2)$ is the energy transferred to the recoiling atom and E_M is the maximum amount that can be transferred in a single collision. This is given in section 2.4.

For the present situation we have $E_M = E_{max} = 4T \frac{M_t M}{(M+M_t)^2}$.

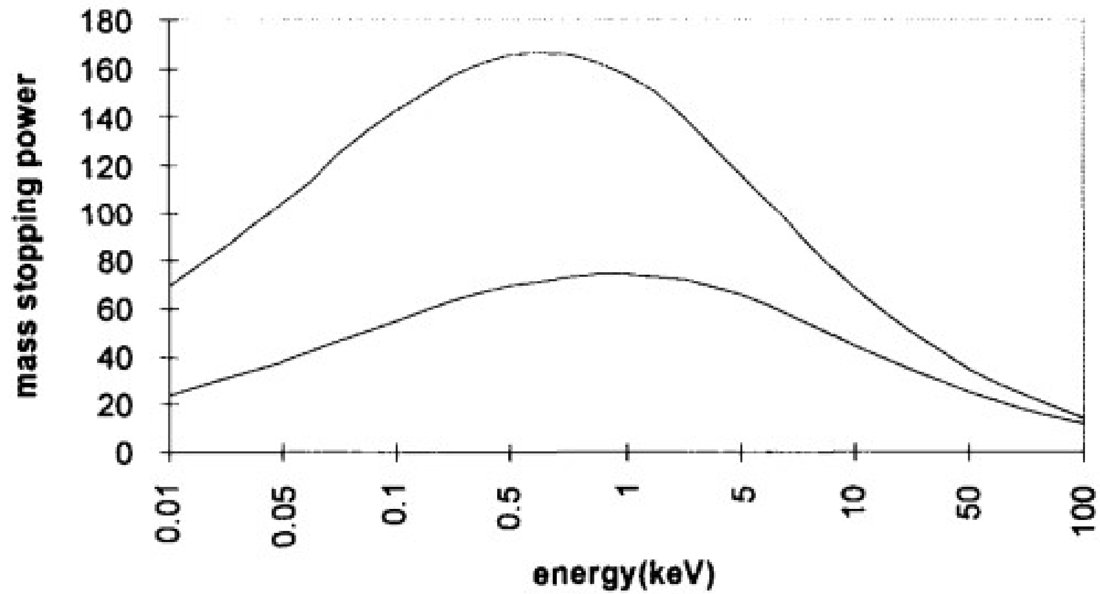

Fig.(2.19) The Mass Nuclear Stopping Power (MeV cm² g⁻¹) due to energy loss of α-particles in carbon (upper curve) and aluminium (lower curve), [3].

2.10 Inelastic Scattering of Light Particles–Radiation Loss

Whenever a charged particle suffers an acceleration – either a deflection of its path or a change in velocity – it radiates electromagnetic energy (*bremsstrahlung*). The wave amplitude of this radiation is proportional to the acceleration times the charge. For a particle mass M and charge ze being accelerated by a charge Ze, the acceleration is proportional to zZe^2/M. The intensity of the emitted radiation is therefore proportional to $(ze \times zZe^2/M)^2$. Consequently, an electron is approximately six orders of magnitude more efficient at producing *bremsstrahlung* radiation than a proton of the same velocity.

Two questions therefore need to be addressed :

(1) What is the intensity of the radiated energy, and
(2) What is the rate of energy loss of the charged particle per unit thickness of material traversed?

The answers to the above come largely from quantum mechanical calculations. These consider the plane wave of an electron which enters a nuclear Coulomb field. The electron is scattered and in the process emits a photon with a small probability.

Expressions for the cross-section depend on the energy region being considered, and on the degree to which the extra-nuclear electrons of an atom tend to screen out the nuclear charge. Different analytical expressions therefore apply to relativistic, non-relativistic, screened and non-screened situations. Since, in addition, there is also the (smaller) possibility of radiation from electron-electron collisions, a full treatment of radiative loss becomes long and involved.

As a result, most theories deal with the Thin Target case. In this a mono-energetic

electron passes through a target so thin that the probability of interaction through any other kind of interaction is negligible. Thus the possibility of ionization, elastic scatter or more than one inelastic scatter, is neglected.

Principle points to note are the following :

- In the classical view, an incoming electron can radiate any fraction of its energy from zero to its total kinetic energy T_o in each collision.
- The quantum mechanical model states that there is a small probability of photon emission at each collision at which a relatively large amount of energy is radiated.
- On average, however, the amounts of energy radiated are approximately the same in both models, even though the spectral distributions are different.
- The momentum of the incoming electron is shared after the collision between the outgoing photon, the recoiling nucleus and the scattered electron. Except at extreme relativistic energies, the momentum of the photon *(hv/c)* is small compared with the electron momentum so it can be emitted in any direction.

The differential cross-section given by quantum mechanics for the emission of *bremsstrahlung* in the energy range $h\nu$ to $h\nu + d(h\nu)$ is [4] :

$$d\sigma_{rad} = \alpha r_e^{\,2} B Z^2 \frac{T + m_0 c^2}{T} \frac{d(h\nu)}{h\nu} \qquad (2.29)$$

Here the electron has initial kinetic energy T and interacts with a nucleus of charge number Z. B is a dimension-less function of Z and T which was evaluated by Heitler [12] and $\alpha = 1/137$ is the Fine Structure Constant. The units of $d\sigma_{rad}$ are cm^2/ nucleus.

The intensity of radiation is given by the product of the cross-section and the amount of energy radiated. Re-arrangement of Eq.(2.29) can therefore be used to define B as the intensity of radiation emitted between $h\nu$ and $h\nu + d(h\nu)$:

$$B = \frac{1}{\alpha r_e^{\,2} Z^2} \frac{T}{\left(T + m_0 c^2\right)} h\nu \frac{d\sigma_{rad}}{d(h\nu)}$$

The total energy loss of an electron as it traverses unit distance in a medium, *dT/dx*, is equal to the energy radiated, $h\nu$, times the probability of this emission. Since this process takes place at all energies as the electron slows down from its initial kinetic energy *T*, we have :

$$\frac{dT}{dx} = N\int_0^T h\nu \, d\sigma_{rad} = N\alpha r_e^{\,2} Z^2 \left(T + m_0c^2\right)\int_0^1 B \, d\left(\frac{h\nu}{T}\right) \tag{2.30}$$

in units J cm^{-1} if there are *N* nuclei cm^{-3}. In Eq.(2.30), the integration limits have changed from 0 to *T* for the electron, to 0 to 1 for the fractional emission $h\nu/T$, [4], [12].

The total cross-section (cm^2/nucleus) is then given by :

$$\sigma_{rad} = \frac{dT}{T + m_0c^2}\,\frac{1}{N\,dx} = \alpha r_e^{\,2} Z^2 \int_0^1 B \, d\left(\frac{h\nu}{T}\right) = \alpha r_e^{\,2} Z^2 \overline{B} \tag{2.31}$$

where the constant $\overline{B}$ is the result of integrating *B* over $h\nu/T$.

The radiative mass stopping power can be obtained from Eq.(2.30) for a target material of atomic number *Z*, atomic weight *A*, density ρ and Avogadro constant N_A. This is :

$$\frac{1}{\rho}\left(\frac{dT}{dx}\right)_{rad} = \frac{N_A \alpha r_e^{\,2}}{A} Z^2 \left(T + m_0c^2\right)\overline{B} \tag{2.32}$$

In Fig.(2.20) note that :

- up to an electron energy of 5MeV, log(normalised energy) ≈ 1, the value of $\overline{B}$ is independent of Z to ~10%. Even at the highest energies the difference between water and lead is less than 20%.
- the correction for the screening of the nuclear Coulomb field by the atomic electrons is greatest for large Z and large energy. It is negligible for all materials at electron energies less than 1 MeV.

Comparisons between the tabulated Radiative Stopping Powers from [5] and the calculated values using Eq.(2.32) can be made in Figs.(2.21) and (2.22). They differ by as much as 20% for certain combinations of T and Z, especially at the two highest electron energies, 5 and 16.2 MeV and the two highest atomic numbers, 42 and 74. There are two reasons which are largely responsible for the differences.

2.10.1 Corrections for the inadequacy of the Born approximation

The Born approximation formed the basis of Eqs.(2.31) and (2.32). However, the approximation is invalid for collisions in which the incoming electron suffers a large change in energy (*e.g.* when $T \sim h\nu$). This can occur either because its initial energy is small or because it is deflected by a large Coulomb potential.

The complete description of electron-nucleus *bremsstrahlung* considers three energy regions :

- T > 50 MeV, where a Coulomb term $f(Z)$ corrects for the assumption that the electron energy before and after the collision is large compared to the electron rest mass.
- T < 2 MeV, where the calculation of the matrix elements cannot be carried out using plane waves for the electron wave functions, but must use exact partial wave function expansions. This procedure requires a numerical solution of the Dirac equation and a numerical evaluation of the *bremsstrahlung* matrix elements.
- 2 MeV < T < 50 MeV, where the cross-section has only a small dependence on both Z and T. In this case, an interpolation procedure between the two extreme energy regions can be used to complete the description.

At high energies electron-electron *bremsstrahlung* begins to become important.

- In this case the recoiling electron can take up a large fraction of the energy and momentum of the incoming electron.
- A total radiative cross-section can be obtained to sufficient accuracy by replacing Z^2 in the Eq.(2.29) by $Z(Z+(\phi_{rad,e}/\phi_{rad,n}))$. The ratio $(\phi_{rad,e}/\phi_{rad,n})$ is a smooth sigmoid function of T and is only weakly dependent on Z. The function plotted in Fig.(2.23) comes from [5].
- The total radiative stopping power is then proportional to $Z^2\phi_{rad,n} + Z\phi_{rad,e}$.