Poisson Brackets:

(1) These are quantities which are invariant under canonical transformations and are the classical analogues of commutation relations between operators in quantum mechanics.

(2) The Poisson bracket of two fns. u, v w.r.t the canonical variables (q, p) is defined as:

$$[u,v]_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$$

Usually the Einstein convention is used and $[u,v]_{q,p} = [u,v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$

(3) If not specifically mentioned, it is understood that the P.B are evaluated w.r.t (q, p)

Some properties:

(1) $[u,v] = -[v,u]$

(2) $[u,u] = [v,v] = 0$

(3) $[cu,v] = [u,cv] = c[u,v]$

for c - ~~cor~~ constant $[cu,v] = \frac{\partial (cu)}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial (cu)}{\partial p}\frac{\partial v}{\partial q}$

(3) $[u+v, \omega] = \frac{\partial (u+v)}{\partial q_i}\frac{\partial \omega}{\partial p_i} - \frac{\partial (u+v)}{\partial p_i}\frac{\partial \omega}{\partial q_i}$

$= \left(\frac{\partial u}{\partial q_i} + \frac{\partial v}{\partial q_i}\right)\frac{\partial \omega}{\partial p_i} - \left(\frac{\partial u}{\partial p_i} + \frac{\partial v}{\partial p_i}\right)\frac{\partial \omega}{\partial q_i}$

$= \frac{\partial u}{\partial q_i}\frac{\partial \omega}{\partial p_i} + \frac{\partial v}{\partial q_i}\frac{\partial \omega}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial \omega}{\partial q_i} - \frac{\partial v}{\partial p_i}\frac{\partial \omega}{\partial q_i}$

Re-arranging terms we get,

$[u+v, \omega] = [u, \omega] + [v, \omega]$ – distributive property.

(4) $[u, v\omega] = [u, v]\,\omega + v\,[u, \omega]$

(5) $\frac{\partial}{\partial t}[u, v] = \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right]$

u

ω $\cancel{v}$

(6) Jacobi's identity:

$[u, [v, \omega]] + [v, [\omega, u]] + [\omega, [u, v]] = 0$

$[u, [v, \omega]] + [\omega, [u, v]] + [v, [\omega, u]] = 0$

(7) $[u, F(\omega_1, \omega_2)] = \cancel{\frac{\partial u}{\partial q_i}}\,\frac{\partial F}{\partial \omega_1}[u, \omega_1] + \frac{\partial F}{\partial \omega_2}[u, \omega_2]$

Proof

$$[u, F(w_1, w_2)] = \frac{\partial u}{\partial q}\frac{\partial}{\partial p}(F(w_1, w_2)) - \frac{\partial u}{\partial p}\frac{\partial}{\partial q}(F(w_1, w_2))$$

$$= \frac{\partial u}{\partial q}\left(\frac{\partial F}{\partial w_1}\frac{\partial w_1}{\partial p} + \frac{\partial F}{\partial w_2}\frac{\partial w_2}{\partial p}\right) - \frac{\partial u}{\partial p}\left(\frac{\partial F}{\partial w_1}\frac{\partial w_1}{\partial q} + \frac{\partial F}{\partial w_2}\frac{\partial w_2}{\partial q}\right)$$

$$= \frac{\partial F}{\partial w_1}\left(\frac{\partial u}{\partial q}\frac{\partial w_1}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial w_1}{\partial q}\right) + \frac{\partial F}{\partial w_2}\left(\frac{\partial u}{\partial q}\frac{\partial w_2}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial w_2}{\partial q}\right)$$

$$= \frac{\partial F}{\partial w_1}[u, w_1] + \frac{\partial F}{\partial w_2}[u, w_2]$$

In general, for $F = (w_1, w_2 --- w_n)$

$$[u, F(w_1, w_2 --- w_n)] = \frac{\partial F}{\partial w_1}[u, w_1] + \frac{\partial F}{\partial w_2}[u, w_2] + \frac{\partial F}{\partial w_3}[u, w_3] + --- + \frac{\partial F}{\partial w_n}[u, w_n]$$

Elementary P.B.

P.B's constructed out of the canonical variables themselves (co-ord & momenta) are called elementary P.B's.

$$[q_i, p_j] = \frac{\partial q_i}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial p_j}{\partial q_i}$$

$\frac{\partial q_i}{\partial q_i} = 1$ & $\frac{\partial p_j}{\partial p_i} = 1$ only if $i = j$; 0

ie $[q_i, p_j] = \delta_{ij}$

Properties:

(1) $[q_i, q_j] = [p_i, p_j] = 0$

(2) $[q_i, p_j] = -[p_j, q_i] = \delta_{ij}$

(3) $[u, q_i] = -\frac{\partial u}{\partial p_i}$ & $[u, p_i] = \frac{\partial u}{\partial q_i}$

(4) $[u, \vec{r}] = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial \vec{r}}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial \vec{r}}{\partial q_i} \right)$ — ①

$\frac{\partial \vec{r}}{\partial p_i} \to 0$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. ($q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$)

Then the 2nd term on R.H.S of ① is:

$$-\frac{\partial u}{\partial p_i}\sum(\hat{i} + \hat{j} + \hat{k}) = -\left(\frac{\partial u}{\partial p_x}\hat{i} + \frac{\partial u}{\partial p_y}\hat{j} + \frac{\partial u}{\partial p_z}\hat{k}\right)$$

$$= -\vec{\nabla}_p u$$

(5) $[u, \vec{p}] = \nabla_r u$.

Hamilton's eqns. in the PB formulation:

(1) For $u = u(q, p, t)$

$$\frac{du}{dt} = \sum_{i=1}^{n} \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} \quad \text{— ①}$$

(2) But from the canonical eqns. we have:

$$\dot{q} = \frac{\partial H}{\partial p} \quad \& \quad -\dot{p} = \frac{\partial H}{\partial q}$$

$\therefore$ (1) becomes:

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial H}{\partial q}\right) + \frac{\partial u}{\partial t}.$$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \quad - (2)$$

If, in (2), $u = q$, (2) becomes

$$\left.\begin{aligned} \dot{q} &= [q, H] \\ \dot{p} &= [p, H] \end{aligned}\right\}$$

& when $u = p$, - Hamiltons equations in P.B formulation

Also, if u is a constant of motion,

~~$\frac{du}{dt} = \frac{\partial u}{\partial t}$~~ or $[u, H] = -\frac{\partial u}{\partial t}$.

<u>Jacobi-Poisson Theorem:</u>

If u & v are any two constants of motion, then, their PB $[u, v]$ is also a constant of motion.

Since ~~If~~ u & v are constants of motion, then,

<u>Given:</u> $[u, H] = -\frac{\partial u}{\partial t}$ or $[H, u] = \frac{\partial u}{\partial t}$ &

$[H, v] = \frac{\partial v}{\partial t}$.

To prove: $[H, [u,v]] = \frac{\partial}{\partial t}[u,v]$

Proof: By Jacobi's identity,

$$[w,[u,v]] + [v,[w,u]] + [u,[v,w]] = 0.$$

Putting $w = H$,

$$\Rightarrow [H,[u,v]] + [v,[H,u]] + [u,[v,H]] = 0$$

or $[H,[u,v]] = -[v,[H,u]] - [u,[v,H]]$

$$= [[H,u],v] + [u,[H,v]]$$

$$= \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right] = \frac{\partial}{\partial t}[u,v]$$

ie if u & v are constants of motion, then their P.B's are also constants of motion.