# Invariance of Poisson Brackets under canonical transformations:

(1) Consider two functions $u(q_i, p_i, t)$ & $v(q, p, t)$.

(2) Let a canonical transformation be made from $(q, p, t) \rightarrow (Q, P, t)$ such that $q = q(Q, P, t)$ & $p = (Q, P, t)$

(3) Correspondingly, let
$u(q, p, t) \longrightarrow u'(Q, P, t)$ &
$v(q, p, t) \longrightarrow v'(Q, P, t)$.

By invariance we mean that
$[u, v]_{q,p} = [u', v']_{Q,P}$.

<u>Proof</u>:
$\frac{\partial u}{\partial q}$ $\approx$ $u(q, p, t) \longrightarrow u'(Q, P, T)$
implies $u(q, p, t) \longrightarrow u(q(Q, P, t), p(Q, P, t), t)$
Ily $v(q, p, t) \longrightarrow v(q(Q, P, t), p(Q, P, t), t)$
Also $Q = Q(q, p, t)$ & $P = (q, p, t)$

$$\therefore \frac{\partial u}{\partial q_i} = \frac{\partial u'}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial u'}{\partial P_j}\frac{\partial P_j}{\partial q_i} \quad - (1)$$

$$\& \ \frac{\partial u}{\partial p_i} = \frac{\partial u'}{\partial Q_j}\frac{\partial Q_j}{\partial p_i} + \frac{\partial u'}{\partial P_j}\frac{\partial P_j}{\partial p_i} \quad - (2)$$

The link between $(q, p)$ and $(Q, P)$ is through the generating function $F$.

Take the $F_1$, $F_2$ type of generating function.

then, $P = -\frac{\partial F_1}{\partial Q}$, $p = \frac{\partial F_1}{\partial q}$ (for $F_1$ type)

(7) Also, $p = \frac{\partial F_2}{\partial q}$, $Q = \frac{\partial F_2}{\partial P}$ for $F_2$ type

$$\Rightarrow p = \frac{\partial F_1}{\partial q} = \frac{\partial F_2}{\partial q}.$$

$$\therefore \frac{\partial P}{\partial q} = \frac{\partial}{\partial q}\left(-\frac{\partial F_1}{\partial Q}\right) = -\frac{\partial^2 F_1}{\partial q \partial Q} = -\frac{\partial}{\partial Q}\left(\frac{\partial F_1}{\partial q}\right)$$

or $\frac{\partial P}{\partial q} = -\frac{\partial p}{\partial Q}$ — (A)

(8) Illy $\frac{\partial Q}{\partial q} = \frac{\partial}{\partial q}\left(\frac{\partial F_2}{\partial P}\right) = \frac{\partial^2 F_2}{\partial q \partial P} = \frac{\partial}{\partial P}\left(\frac{\partial F_2}{\partial q}\right)$

or $\frac{\partial Q}{\partial q} = \frac{\partial p}{\partial P}$ — (B)

(9) Illy $\frac{\partial P}{\partial p} = -\frac{\partial q}{\partial Q}$ — (C) & $\frac{\partial q}{\partial Q} = \frac{\partial P}{\partial p}$ — (D)

(10) Using (A) & (B) in (1),

$$\frac{\partial u}{\partial q} = \frac{\partial u'}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial u'}{\partial P}\frac{\partial p}{\partial Q} \quad — (3)$$

(11) Using (C) & (D) in (2),

$$\frac{\partial u}{\partial p} = -\frac{\partial u'}{\partial Q}\frac{\partial q}{\partial P} + \frac{\partial u'}{\partial P}\frac{\partial q}{\partial Q} \quad - (4)$$

(12) $$[u,v]_{(q,p)} = \frac{\partial u}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q} \quad - (5)$$

(13) Multiplying (3) by $\frac{\partial v}{\partial p}$ & (4) by $\frac{\partial v}{\partial q}$, we get,

$$\left(\frac{\partial u'}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial u'}{\partial P}\frac{\partial p}{\partial Q}\right)\frac{\partial v}{\partial p} - \left(-\frac{\partial u'}{\partial Q}\frac{\partial q}{\partial P} + \frac{\partial u'}{\partial P}\frac{\partial q}{\partial Q}\right)\frac{\partial v}{\partial q}$$

$$= \frac{\partial u'}{\partial Q}\frac{\partial p}{\partial P}\frac{\partial v}{\partial p} - \frac{\partial u'}{\partial P}\frac{\partial p}{\partial Q}\frac{\partial v}{\partial p} + \frac{\partial u'}{\partial Q}\frac{\partial q}{\partial P}\frac{\partial v}{\partial q} - \frac{\partial u'}{\partial P}\frac{\partial q}{\partial Q}\frac{\partial v}{\partial q}$$

$\cancel{= \frac{\partial v'}{\partial Q} = \left(\frac{\partial v}{\partial q}\right.}$ ~~But~~ $\cancel{\frac{\partial u'}{\partial Q} = (\partial}$

$$= \frac{\partial u'}{\partial Q}\left(\frac{\partial p}{\partial P}\frac{\partial v}{\partial p} + \frac{\partial q}{\partial P}\frac{\partial v}{\partial q}\right) - \frac{\partial u'}{\partial P}\left(\frac{\partial p}{\partial Q}\frac{\partial v}{\partial p} + \frac{\partial q}{\partial Q}\frac{\partial v}{\partial q}\right) \quad - (6)$$

(14) But $\because v' = v'(Q,P,t) = v(q(Q,P,t), p(Q,P,t), t)$

$\therefore \frac{\partial v'}{\partial P} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial P} \rightarrow$ the first term in bracket of (6) $\quad - (7)$

& $\frac{\partial v'}{\partial Q} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial Q} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial Q} \quad - (8) \rightarrow$ 2nd term in the bracket of (6) i.e, eqn. 6 becomes:

$$[u,v]_{(q,p)} = \frac{\partial u'}{\partial Q_j}\frac{\partial v'}{\partial P_j} - \frac{\partial u'}{\partial P_j}\frac{\partial v'}{\partial Q_j} \quad - (9)$$

$$= [u', v']_{(Q,P)}$$

ie the Poisson Brackets are invariable under a canonical transformation.

PB's involving Angular momentum.

(1) Starting with the definition of $\vec{L} = \vec{r} \times \vec{p}$,

$$L_x = y p_z - z p_y$$

$$L_y = z p_x - x p_z$$

$$L_z = x p_y - y p_x$$

$(q_1 = x,\ q_2 = y,\ q_3 = z \ \&\ p_1 = p_x,\ p_2 = p_y \ \&\ p_3 = p_z)$

$$[L_x, L_y] = \sum_{i=1}^{3} [(y p_z - z p_y), (z p_x - x p_z)]$$

$$= \frac{\partial}{\partial x}(y p_z - z p_y)\frac{\partial}{\partial p_x}(z p_x - x p_z)$$

$$- \frac{\partial}{\partial p_x}(y p_z - z p_y)\frac{\partial}{\partial x}(z p_x - x p_z)$$

$$+ \frac{\partial}{\partial y}(y p_z - z p_y)\frac{\partial}{\partial p_y}(z p_x - x p_z)$$

$$- \frac{\partial}{\partial p_y}(y p_z - z p_y)\frac{\partial}{\partial y}(z p_x - x p_z)$$

$$+\frac{\partial}{\partial z}(yp_z - zp_y)\frac{\partial}{\partial p_z}(zp_x - xp_z)$$

$$-\frac{\partial}{\partial p_z}(yp_z - zp_y)\frac{\partial}{\partial z}(zp_x - xp_z)$$

$$= (0 - p_y)(0 - x) - (y - 0)(p_x - 0)$$

$$= xp_y - yp_x = L_z$$

or $[L_x, L_y] = L_z$ }

$\epsilon_{ijk} = 1$ for cyclic $i, j, k$
$= -1$
$= 0$ if $k = i = j$

or $[L_i, L_j] = \epsilon_{ijk} L_k$

The Poisson Bracket $\longrightarrow$ $i\hbar$ Commutator Bracket
Classical $\longrightarrow$ Quantum

{ In QM $L_x$, $L_y$ and $L_z$ are defined as components of angular momentum if $[L_x, L_y] = i\hbar L_z$ in cyclic order.