Permutation group.

Definition: A permutation of a set A is nothing but a bijection from A to itself.

Or we can also state as: Let f is a function defined from a set A to A itself, then f is called a permutation of A if

(a) f is an injection function i.e. f is one-one
(b) f is a surjection function i.e. f is onto.

$\Rightarrow$ f is one-one onto.

Two row notation form: Let A is a finite set. i.e. $A = (a_1, a_2, a_3, \ldots a_n)$ of n distinct elements and $f : A \to A$ is one-one onto. In two row form

$$\Rightarrow f = \begin{pmatrix} a_1 & a_2 & a_3 \cdots & a_n \\ f(a_1) & f(a_2) & f(a_3) \cdots & f(a_n) \end{pmatrix}$$

i.e. $f(a_1)$ is image of a etc.

Ex $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ Here $f(1) = 2$
$f(2) = 4$ etc.

Here f is one-one onto

$\Rightarrow$ f is a permutation defined on (1, 2, 3, 4)

Here $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 1 & 3 & 2 \end{pmatrix}$

Because in both $f(1) = 2$; $f(2) = 4$, $f(3) = 1$ & $f(4) = 3$

Ex Let $A = \{1, -1, i, -i\}$ & $f(x) = ix$

Then $f(1) = i$, $f(-1) = -i$

$f(i) = -1$, $f(-i) = 1$

$\Rightarrow f = \begin{pmatrix} 1 & -1 & i & -i \\ i & -i & -1 & 1 \end{pmatrix}$

Here f is one-one onto $\Rightarrow f$ is a permutation

Ex Similarly Identity function i.e.

$I(x) = x \ \forall x \in A$, then I is again an identity permutation.

Definition: Group of permutations;

(i) let $A = (a, b)$ then two permutations can be written for A.

Like $\sigma_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$; $\sigma_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$\Rightarrow$ Set of permutation defined on A i.e. S_A or $S_{(a,b)}$

$\Rightarrow S_A = \{\sigma_1, \sigma_2\}$

(ii) let $A = (a, b, c)$ then $\underline{|3}$ permutations can be written for A.

Like $\sigma_1 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$, $\sigma_4 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$

$\sigma_5 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$, $\sigma_6 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$

$\Rightarrow S_A = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$

$\Rightarrow S_A$ contains $\underline{|3} = 6$ members.

$\Rightarrow$ If A contains n elements then S_A contains $\underline{|n}$ elements i.e. $\underline{|n}$ permutations.

Composition of permutations:

Ex: Let us take $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ are two permutations defined on the set $A = (1, 2, 3, 4)$

Basically f & g are one-one onto functions so we can define composition of functions fog & gof.

Let us find

$$fog(1) = f[g(1)] = f(2) = 4$$
$$fog(2) = f[g(2)] = f(3) = 1$$
$$fog(3) = f[g(3)] = f(4) = 3$$
$$fog(4) = f[g(4)] = f(1) = 2$$

$$\Rightarrow fog = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

It can be obtain arrange the first row of first permutation & second row of second permutation is a similar order. then. Result permutation will be the First row of second permutation & second row of first permutation ∴ It can also be done as follows.

$$fog = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} \boxed{1 \; 2 \; 3 \; 4} \\ 2 \; 4 \; 1 \; 3 \end{pmatrix} \begin{pmatrix} 4 \; 1 \; 2 \; 3 \\ \boxed{1 \; 2 \; 3 \; 4} \end{pmatrix}$$

$$\text{Result} = \begin{pmatrix} 4 & 1 & 2 & 3 \\ 2 & 4 & 1 & 3 \end{pmatrix} \quad \underline{\text{Ans}} \text{ ——— (i)}$$

This Result is also interpreted as product of f and g permutations.

Similarly $g\circ f$ can be obtained.

$$g\circ f = \begin{pmatrix} \boxed{1 & 2 & 3 & 4} \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 2 & 3 & 4 & 1 \\ \boxed{4 & 1 & 3 & 2} \end{pmatrix}$$

To get the Result, make the first Row of g & second Row of f in the same order or we can say that arrange the elements of second Row of f according to first Row of g.

$$g\circ f = \begin{pmatrix} \boxed{1 & 2 & 3 & 4} \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 3 & 1 & 4 & 2 \\ \boxed{1 & 2 & 3 & 4} \end{pmatrix}$$

$$\Rightarrow g\circ f = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix} \text{ ——— (ii)}$$

From (i) & (ii) we can interpret that

$f\circ g \neq g\circ f$ in general.

$\Rightarrow$ Product of permutation is not commute in general.

i.e: $\sigma_1 \circ \sigma_2 \neq \sigma_2 \circ \sigma_1$ in general.

Identity permutation:

Similarly, we can define Identity permutation by the function on A.

$$I(x) = x \quad \forall x \in A$$

So $I\circ f = f\circ I = f$

Theorem:- Let f and g are two permutations defined on a set A, then $f\circ g$ & $g\circ f$ ~~are~~ will also be the permutations.

Definition : Symmetric Group !

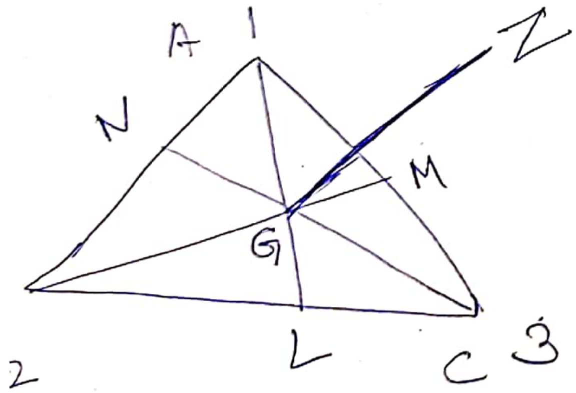
Let $A = \{1, 2, 3 \ldots n\}$, then the group of permutations on the set A is called symmetric group of degree n and it is denoted by S_n. This symmetric group S_n will be of order $\lfloor n$.

Ex S_3 = contains six elements like
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \text{ etc.}$$

Let us denote all these six as
$S_3 = \{p_0, p_1, p_2, \mu_1, \mu_2, \mu_3\}$

It is also ~~classi~~ classified that S_3 is a group of symmetries of an equilateral triangle.

The members of S_3 can be obtained by the change positions of the vertices of the triangle by rotation along GZ, BM, CN & AL by $120^\circ, 240^\circ$ ~~360~~ 360°, 180°, 180°, & 180° respectively.



$\Rightarrow$ By rotation, we can have symmetries.