# Hamilton-Jacobi Theory

(1) Suppose a system is identified by a Hamiltonian $H(q_1, q_2 \dots q_n, p_1, p_2 \dots p_n, t)$

(2) Now if, suppose all $q_i$ are cyclic,
$\Rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i} = 0$ or the $p_i$'s are all constants. So $H = H(\alpha_1, \alpha_2 \dots \alpha_n)$ where, $\alpha_i = p_i$ (a constant) & $H$ is a constant

(3) $\Longrightarrow \quad \dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \text{constant} = \omega_i(\alpha)$

$\therefore q_i = \omega_i t + \text{constant}$. Or, the equations of motion are linear in ~~nature~~ time.

(4) So, if we can make a canonical transformation such that
$(q, p) \longrightarrow (Q, P)$ & $H \longrightarrow K$, which makes as many of the co-ordinates cyclic as possible, then ~~l~~it is easier to arrive at a solution.

(5) A proper set of $(P, Q)$ can be had for which $K$ is zero.

(6) K is related to H through a generating function F through:

$$K = H + \frac{\partial F}{\partial t} = 0 \text{ for choice of } K=0.$$

(7) Let $F = F(q, P, t) \Rightarrow F = F_2(q, P, t)$.

For the $F_2$ function then, $p_i = \frac{\partial F_2}{\partial q_i}$ & $Q_i = \frac{\partial F_2}{\partial P_i}$

$\therefore H(q_1, q_2 \ldots q_n, p_1, p_2 \ldots p_n, t)$

becomes $H\left(q_1, q_2 \ldots q_n, \frac{\partial F_2}{\partial q_1}, \frac{\partial F_2}{\partial q_2} \ldots \frac{\partial F_2}{\partial q_n}, t\right)$

(8) Thus $\because K = 0$

$$H\left(q_1, q_2 \ldots q_n, \frac{\partial F_2}{\partial q_1}, \frac{\partial F_2}{\partial q_2} \ldots \frac{\partial F_2}{\partial q_n}, t\right) + \frac{\partial F_2}{\partial t} = 0 \quad \text{— (1)}$$

(1) is called the Hamilton-Jacobi equation.

(9) The solution to the H-J equation is denoted by S and is called Hamilton's principal function

(10) So there are 1st order ~~of~~ differential equations

(11) So if $S = S(q_1, q_2 \dots q_n, \alpha_1, \alpha_2 \dots \alpha_n, t)$
$\alpha_1, \alpha_2 \dots$ e.t.c are independent constants of integration.

(12) ~~The~~ A ~~can~~ Canonical transformation in which $(q, p, t) \to (q, P, t)$ is mediated by the $F_2$ ~~function~~ type of generating function.

$$\therefore \alpha_i = P_i.$$

(13) For the ~~$F_2$ typ~~ $F_2(q, P, t)$ type of fn;

$$p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} \; \& \; K = H + \frac{\partial F_2}{\partial t}$$

(derived earlier while doing CT).

(14) $\therefore$ in this case, $p_i = \dfrac{\partial S(q, \alpha, t)}{\partial q_i}$ — ①

(15) At time $t = t_0$, we get '$n$' equations which relate the $n$ $\alpha$'s with initial values of $q$ & $p$.

(16) We can thus obtain $\alpha$'s, the constants of integration depending on the initial conditions of the given problem.

(17) $\because$ $Q_i = \dfrac{\partial S(q, \alpha, t)}{\partial \alpha_i} = \beta_i$ a set of constants. This can similarly be obtained from the initial ~~canad~~ conditions at $t = t_0$ ~~if kno~~ (i.e, known initial values of $q_i$)

(18) or, $q_r = q_r(\alpha, \beta, t)$ & $p_i = p_i(\alpha, \beta, t)$.

## Physical Significance of S.

For,

(1) $S = S(q, \alpha, t)$

consider the total time derivative of S,

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial \alpha_i}\dot{\alpha}_i + \frac{\partial S}{\partial t}$$

But since $\alpha_i$'s are constants w.r.t 't',

$\dot{\alpha}_i = 0$

$$\& \quad \frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}.$$

Also, $\because$ $p_i = \dfrac{\partial S}{\partial q_i}$, & $K = H + \dfrac{\partial S}{\partial t}$

$\cancel{\dfrac{dS}{dt} = p_i \dot{q}_i}$ $\therefore$

(2) $\because$ we have chosen $K=0$,

$$\Rightarrow \quad H + \frac{\partial S}{\partial t} = 0 \text{ or } \frac{\partial S}{\partial t} = -H.$$

(3) $\therefore$ $$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} = p_i\dot{q}_i - H = L.$$

or, $S = \int L\,dt.$

which is the action integral.