Defⁿ:- cyclic permutation: Let A be a set and σ is a permutation defined on A. Then, σ is said to be cyclic if there exists a finite subset $\{a_1, a_2, a_3 \dots a_r\}$ of A such that

$$\sigma(a_1) = a_2, \quad \sigma(a_2) = a_3, \quad \sigma(a_3) = a_4$$

$$\sigma(a_r) = a_1 \text{ and } \sigma(x) = x \text{ if } x \in A \text{ but } \notin \{a_1, a_2, \dots a_r\}$$

If σ is a cyclic permutation, then we denote it as

$$\sigma = (a_1, a_2, a_3, \dots a_r) \text{ ——— } (1)$$

it means every element is followed by it's image and image of last element is the first element. The total number of elements which appear in such type of representation i.e. (1), is called length of σ. This is called a single row representation of σ.

<u>for example</u> Let $\sigma = (2\ 3\ 5\ 7)$ and σ is defined on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ i.e. $A = \{1, 2, \dots 8\}$.

then we denote $\sigma = (2\ 3\ 5\ 7) \in A_8$ or S_8

$$\Rightarrow \sigma(1) = 1, \ \sigma(2) = 3, \ \sigma(3) = 5, \ \sigma(4) = 4,$$

$$\sigma(5) = 7, \ \sigma(6) = 6, \ \sigma(7) = 2, \ \sigma(8) = 8$$

we can also be represented in single ~~row~~ in two row notation form, as follows.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 & 8 \end{pmatrix}$$

Here length of $\sigma = 4$

property: $(2\ 3\ 5\ 7)$ can also represented as

$(3\ 5\ 7\ 2) = (5\ 7\ 2\ 3) = (7\ 2\ 3\ 5)$

Here $\sigma(2) = 3$ Again $\sigma(\sigma(2)) = \sigma(3) = 5$

$\Rightarrow \sigma^2(2) = 5$

Similarly $\sigma^3(2) = \sigma(5) = 7$

$\sigma^4(2) = \sigma(7) = 2$

$\Rightarrow \sigma^4(2) = 2$

Similarly $\sigma^4(3) = 3$, $\sigma^4(5) = 5$, $\sigma^4(7) = 7$

$\Rightarrow \sigma^4(x) = x \ \forall x \in A_8$

$\Rightarrow$ order of $\sigma = 4$

$\Rightarrow$ length and order of a cyclic permutation is same.

Definition:- Two cycles $(i_1, i_2, i_3 \ldots i_r)$ and $(j_1, j_2, \ldots j_s)$ defined on A_n are said to be disjoint cycles if no element is common between these two.

i.e. $(i_1, i_2, \ldots i_r) \cap (j_1, j_2, \ldots j_s) = \phi$.

Ex $(1\ 2\ 4), (3\ 5, 7)$ are disjoint cycles.

Th The product of two disjoint cycles is commutative

Example Let $\sigma = (1\ 6\ 7) \in S_7$ and $\tau = (2\ 4, 5) \in S_7$, then find $\sigma\tau$ and $\tau\sigma$

Sol^n. Given $\sigma = (1\ 6\ 7) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 4 & 5 & 7 & 1 \end{pmatrix}$ ——(i)

and $\tau = (2\ 4\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 5 & 2 & 6 & 7 \end{pmatrix}$ ——(ii)

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 4 & 5 & 7 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 5 & 2 & 6 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{1\ 2\ 3\ 4\ 5\ 6\ 7} \\ 6\ 2\ 3\ 4\ 5\ 7\ 1 \end{pmatrix} \begin{pmatrix} 1\ 5\ 3\ 2\ 4\ 6\ 7 \\ \boxed{1\ 2\ 3\ 4\ 5\ 6\ 7} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 & 3 & 2 & 4 & 6 & 7 \\ 6 & 2 & 3 & 4 & 5 & 7 & 1 \end{pmatrix} \text{——(iii)}$$

can also be written as

$$= (1\ 6\ 7)(2\ 4\ 5)(3)$$

Again.

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 5 & 2 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 4 & 5 & 7 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{1\ 2\ 3\ 4\ 5\ 6\ 7} \\ 1\ 4\ 3\ 5\ 2\ 6\ 7 \end{pmatrix} \begin{pmatrix} 7\ 2\ 3\ 4\ 5\ 1\ 6 \\ \boxed{1\ 2\ 3\ 4\ 5\ 6\ 7} \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 2 & 3 & 4 & 5 & 1 & 6 \\ 1 & 4 & 3 & 5 & 2 & 6 & 7 \end{pmatrix} = (7\ 1\ 6)(2\ 4\ 5)$$

$$= (1\ 6\ 7)(2\ 4\ 5) \text{——(iv)}$$

From (iii) & (iv) $\sigma\tau = \tau\sigma \Rightarrow$ Product is commutative.

Transposition

Definition: A cyclic permutation of length two is called transposition.

$\Rightarrow$ $\sigma = (i\ j)$ is a transposition such that

$\sigma(i) = j$, $\sigma(j) = i$ and $\sigma(x) = x$ if $x \neq i, j$

Length of $\sigma = 2$ $\quad \because \sigma^2(i) = i$

In this case $\quad \sigma^{-1}(\sigma^2(i)) = \sigma^{-1}(i)$

$\Rightarrow (\sigma^{-1}\sigma)\sigma(i) = \sigma^{-1}(i)$

$\Rightarrow I(\sigma(i)) = \sigma^{-1}(i)$

$\Rightarrow \sigma(i) = \sigma^{-1}(i)$

$\Rightarrow$ Every transposition is it's own inverse.

Let us take $\sigma = (i_1, i_2, i_3 \ldots i_\ell)$ is the cycle of length ℓ, then

$\sigma = (i_1, i_2, \ldots i_\ell) = (i_1, i_2)(i_1\ i_3)(i_1\ i_4) \ldots (i_1\ i_\ell)$

$\Rightarrow$ Every cycle of length ℓ can be expressed as product of $\ell - 1$ transpositions.

Example: Express $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 2 & 5 \end{pmatrix}$ as product of transpositions.

Sol. Here $\sigma = (1\,3\,4)(2\,6\,5)$
$= (1\,3)(1\,4)(2\,6)(2\,5)$

It can also be written as
$\sigma = (1\,3)(1\,4)(1\,2)(1\,6)(1\,2)(1\,5)$

Even and odd permutation :-

Inversion :- Let $\sigma \in S_n$ i.e. permutation defined on $(1\,2\,3\,4\,5\,6\,7 \ldots n)$, then the pair $(i\,j)$ where $0 < i < j \leq n$, is called an inversion for σ if $\sigma(i) > \sigma(j)$ Here $0 < i < j \leq n$.

Signature of σ : The total number of inversions for a permutation is called signature of σ and we write it as $\mathrm{Sig}(\sigma)$

Def. A permutation σ is called an even permutation if, total number of inversions for σ is even or $\mathrm{Sig}(\sigma) =$ even.
Similarly $\mathrm{Sig}(\sigma) =$ odd, then σ is called odd permutation.

Ex $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 6 & 1 \end{pmatrix} \in S_6$,
then, Here $(2\,1), (5\,4)(5\,3)(5\,1)(4\,3)$ $(4\,1)(3\,1)$ are the inversions

$\Rightarrow \mathrm{Sig}_n(\sigma) = 7 \Rightarrow$ Permutation σ is odd.