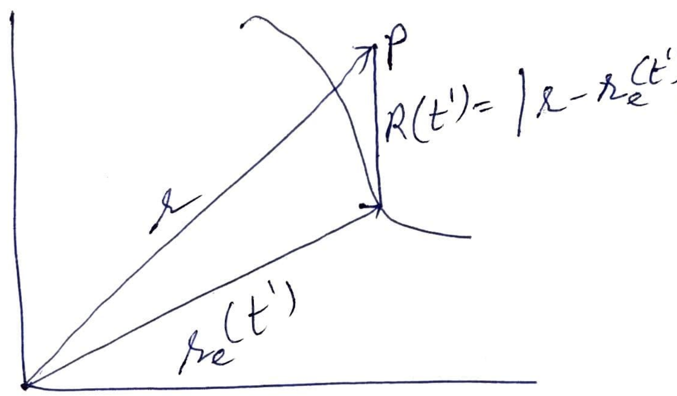
Lienard - Wiechart Potentials

We consider the application of the retarded potentials to compute the radiation from a single charged particle, say, an electron, in arbitrary motion. The calculation of the potentials depend upon the position and velocity of the charge at the retarded time $t - \frac{|r-r'|}{c}$, we must know the details of the motion of the charge.

In figure we show a trajectory of the electron described by the radius vector $r_e(t')$.



The calculation of the potentials as given in the formulae

$$\phi(r,t) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho\left(r', t - \frac{|r-r'|}{c}\right)}{|r-r'|} d\tau$$

$$A(r,t) = \frac{\mu_0}{4\pi} \int_V \frac{j\left(r', t - \frac{|r-r'|}{c}\right)}{|r-r'|} d\tau$$

involves a retarded time integration over the entire volume containing the charge.

We assume that the electron has a finite radius, but shall consider only those properties which are independent of the magnitude of the radius. For an electron we may express the retarded potential in terms of a delta function.

$$\phi(r,t) = \frac{e}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{\delta\left\{t' - \left(t - \frac{|r - r_e(t')|}{c}\right)\right\}}{|r - r_e|} dt' \quad \text{--- (1)}$$

If we put the integral in the form $\int_{-\infty}^{\infty} f(x)\,\delta(x-x')\,dx$, following a property of delta-function, its integral is readily found and is equal to $f(x')$.

We introduce a new variable t'' such that

$$t'' = t' - t + \frac{|r - r_e(t')|}{c} \quad \text{--- (2)}$$

$$dt'' = dt' + \frac{1}{c}\frac{d}{dt'}|r - r_e(t')|\,dt' \quad \text{--- (3)}$$

We have taken $dt = 0$ because the observations made at a fixed time t.

Let the coordinates of the fixed point p be x_1, x_2, x_3 and those of the electron $x_{e_1}(t')$, $x_{e_2}(t')$, $x_{e_3}(t')$.

Now
$$|r - r_e(t')| = \sqrt{\sum_i \left\{x_i - x_{e_i}(t')\right\}^2} \quad \text{--- (4)}$$

Therefore

$$\frac{1}{c}\frac{d}{dt'}|r - r_e(t')| = \frac{1}{c}\sum_i \frac{\partial}{\partial x_{e_i}}|r - r_e(t')|\frac{dx_{e_i}}{dt'} \quad \text{--- (5)}$$

Since $\frac{\partial}{\partial x_{e_i}}|r - r_e(t')|$ are the components of the gradient of $|r - r_e(t')|$ and $\frac{\partial x_e}{dt'}$ are the components of $\frac{dr_e}{dt}$ we can write

$$\frac{1}{c}\frac{d}{dt'}\left|r - r_e(t')\right| = \frac{1}{c}\, \text{grad}_{r_e}\left|r - r_e(t')\right| \cdot \frac{dr_e}{dt} \quad \text{---(6)}$$

$$\text{grad}_{r_e}\left|r - r_e(t')\right| = -\frac{r - r_e(t')}{\left|r - r_e(t')\right|} = -\frac{R}{|R|}$$

where $\underline{r - r_e(t) = R}$ --(7)

Now $\frac{dr_e}{dt} = u$ (velocity of the electron)

$$\Rightarrow \frac{1}{c}\frac{d}{dt'}\left|r - r_e(t')\right| = -\frac{1}{c}\frac{R}{|R|}\cdot u = -\frac{\beta \cdot R}{|R|} \quad \text{-(8)}$$

where $\beta = \frac{u}{c}$

$$\Rightarrow dt'' = dt'\left[1 - \frac{\beta \cdot R}{|R|}\right]$$

i.e. $$dt' = \frac{|R|}{|R| - \beta \cdot R}\, dt'' \quad \text{--(9)}$$

Hence the potential (eq^n (1)) can be expressed as

$$\phi(r,t) = \frac{e}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{\delta(t'')}{|R(t')|}\, \frac{|R(t')|}{|R(t')| - \beta(t')\cdot R(t')|}\, dt'' \quad \text{---(10)}$$

$$= \frac{e}{4\pi\varepsilon_0}\left[\frac{1}{|R(t')| - \beta(t')\cdot R(t')}\right]_{t''=0} \quad \text{---(11)}$$

Since $t''=0$ implies $t' = t - R(t')/c$

$$\phi(r,t) = \frac{e}{4\pi\varepsilon_0}\, \frac{1}{|R - \beta\cdot R|}\bigg|_{t' = t - \frac{R(t')}{c}} \quad \text{---(12)}$$

Similarly $$A(r,t) = \frac{\mu_0 e}{4\pi}\left[\frac{u}{R - \beta\cdot R}\right]_{t' = t - \frac{R(t')}{c}} \quad \text{---(13)}$$

The potentials eq^n (12) and (13) are called Lienard-Wiechart Potentials. The are dependent on velocity of the electron but independent of the extent of the charge.