Vector Potential and Poisson's Equation

The scalar potential is not useful for analysis of magnetic field except the condition that current is flowing in single loop or no current distribution. The necessary condition for establishment of scalar potential, curl $\vec{B} = 0$ is not satisfied generally. And there is no unique way to define the scalar potential.

The existence of vector potential depends on the eq^n div $\vec{B} = 0$ or $\nabla \cdot \vec{B} = 0$ --- (1)

We know that

$$\text{div curl } \vec{A} = 0$$

where $\vec{A} \rightarrow$ Arbitrary vector

We can, therefore express B as curl of a vector A

$$\vec{B} = \text{curl } \vec{A} \quad --- (2)$$

The vector $\vec{A}$ which satisfies eq^n(2) is known as vector potential.

From eq^n (2), eq^n $\vec{\nabla} \cdot \vec{B} = 0$ is automatically satisfied.

Now we should select $\vec{A}$ in such a way that eq^n curl $\vec{B} = \mu_0 \vec{J}$ is also satisfied.

$$\text{curl curl } \vec{A} = \mu_0 \vec{J}$$

from vector identity

$$\text{curl curl } \vec{A} = \text{grad div } \vec{A} - \nabla^2 \vec{A}$$

where $\vec{A}$ is arbitrary vector

Now, we let $\operatorname{div}\vec{A}=0$; $\nabla\cdot\vec{A}=0$ --- (3)

From eqn (3) we find

$$\nabla^2\vec{A} = -\mu_0 j \quad --- (4)$$

Eqn (4) is poisson's eqn for vector potential in terms of current density. The general solution of eqn (4) for unbounded region is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{r}\, dV \quad --- (5)$$

One can easily see that A is not uniquely defined by eqn (2). For, we could add to A any function whose curl is zero, say, gradient of a scalar ψ, and still have the same B.

$$\nabla\times(A+\nabla\psi) = \nabla\times A + \nabla\times\nabla\psi$$
$$= \nabla\times A = B$$

In electrostatics the scalar potential V was not completely specified by its definition $E=-\nabla V$. If V is the potential for some problem, a different potential $V'=V+C$, where C is a constant, gives the same field

$$-\nabla V' = -\nabla(V+C) = -\nabla V - \nabla C$$
$$= -\nabla V = E$$

The $\vec{A} = \vec{A} + \operatorname{grad}\psi$ → can be called Gauge transformation

$B=\nabla\times A$ and $\nabla\cdot A=0$ together define the vector potential A which satisfies the fundamental equation $\nabla\cdot B=0$.

Solutions of Poisson's equation for line surface currents

The Poisson eq^n of vector potential

$$\nabla^2 \vec{A} = -\mu_0 \vec{j}$$

$j \rightarrow$ current density

The sol^n of this eq^n for vector potential $\vec{A}$ is

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \nabla \times \left(\frac{\vec{m}}{r} \right) \quad \text{---} (1)$$

$\vec{m} \rightarrow$ is the dipole moment of current loop.

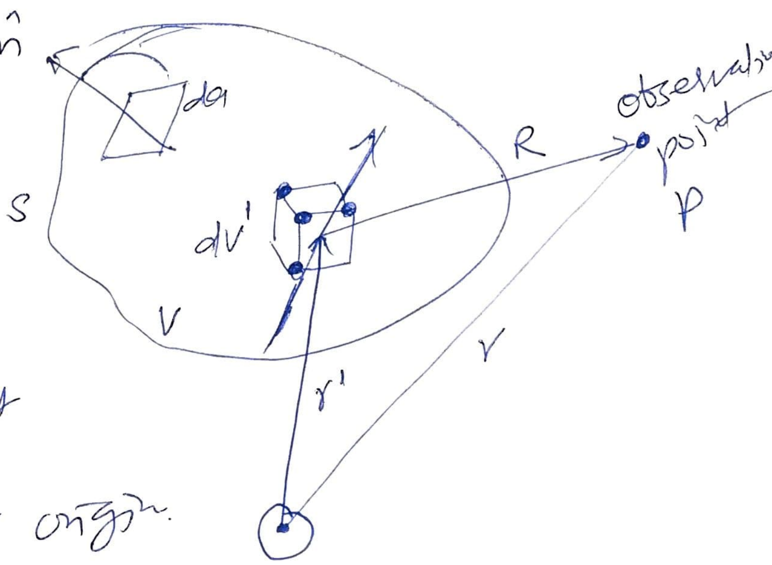
eq^n (1) can be written as

$$\vec{A}(r) = -\frac{\mu_0}{4\pi} \vec{m} \times \nabla\left(\frac{1}{r}\right) \quad \text{---} (2)$$

eq^n (2) is true for any orientation of $\vec{m}$.

We take volume V, there are many magnetic dipoles ~~inside~~ it due to current distribution.

We need to find potential $\vec{A}(r)$ at point P due to volume dv'. dv' is at distance r' from origin and P point is at distance r from origin.



$\vec{A}(r)$ could be obtained from eq^n (2).

If $r \gg r'$, then r could be replaced by R in eq^n (2) where R is distance of dv' from point P.

$$\vec{A}(r) = -\frac{\mu_0}{4\pi} \vec{m} \times \nabla\left(\frac{1}{R}\right)$$

$$= \frac{\mu_0}{4\pi} \vec{m} \times \nabla'\left(\frac{1}{R}\right)$$

here $\nabla' = -\nabla$;' $\vec{r}' = \vec{r} - \vec{R}$

If magnetic moment of unit volume is $\vec{m}(r')$, then magnetic moment of volume dV' at distance r' is $\vec{m}(\vec{r}') dV'$ => $\vec{m} = m(\vec{r}') dV'$

$$\text{so } \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{m} \times \nabla'\left(\frac{1}{R}\right) dV' \quad \text{--- (3)}$$

from vector Identity

$$\nabla \times (a\vec{b}) = \nabla a \times b + a \nabla \times b \quad \text{--- (4)}$$

where $a \to$ constant

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\int_V \frac{\nabla' \times \vec{m}}{R} dV' - \int \nabla' \times \left(\frac{m}{R}\right) dV'\right]$$

$$\oint_S \hat{n} \times \vec{b}\, ds = \int_V \nabla' \times \vec{b}\, dV \quad \text{--- (5)}$$

$$\text{So } \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\int_V \frac{\nabla' \times \vec{m}}{R} dV' + \oint_S \frac{\vec{m} \times \hat{n}}{R} ds'\right] \quad \text{--- (6)}$$

According to eqn (6) the volume distribution of magnetic dipoles is equivalent to a surface current density and a volume current density

$$J_s = \vec{m} \times \hat{n} \quad \text{--- (7)}$$

$$\text{and } J_V = \vec{\nabla} \times \vec{m} \quad \text{--- (8)}$$